

11.5 Exponential and Logarithmic Equations

In this section we want to solve equations that contain exponential and logarithmic functions. Let's start by reviewing some properties.

Properties of Exponential and Logarithmic Equations

Let $a, x,$ and y be real numbers such that $a, x, y > 0, a \neq 1$. Then the following are true.

1. $a^x = a^y$ if and only if $x = y$
2. $\log_a x = \log_a y$ if and only if $x = y$
3. $\log_a a^x = x, \ln e^x = x$
4. $a^{\log_a x} = x, e^{\ln x} = x$

Recall that the last two properties are true because the exponential and logarithmic functions are inverse functions.

We have used these properties in class to solve simpler equations. We want to do more complicated equations now. Before we do so, here is the procedure that we will use.

Solving Exponential and Logarithmic Equations

1. To solve an exponential equation, isolate the exponential expression, then take the logarithm of both sides of the equation (using a suitable base) and solve for the variable.
2. To solve a logarithmic equation, isolate the logarithmic expression, then exponentiate both sides of the equation (using a suitable base) and solve for the variable.

It is important to note that when we solve a logarithmic equation, we must check our answer. The reason for this is that the logarithm has a domain restricted to only positive real numbers. Hence, we need to check for zeros and negatives in the logarithm.

Let's proceed to our equations. We will review a few easier ones first.

Example 1

Solve $5 + e^{x+1} = 20$.

Solution:

This is an exponential equation. Thus we must isolate the exponential expression.

$$5 + e^{x+1} = 20$$

$$e^{x+1} = 15$$

Now let's take the logarithm of both sides. The base we should use is e , since $\ln e^x = x$. Thus, we get

$$e^{x+1} = 15$$

$$\ln e^{x+1} = \ln 15$$

$$x + 1 = \ln 15$$

Now just solve for x .

$$x = \ln 15 - 1$$

So the solution is $x = \ln 15 - 1 \approx 1.708$.

Example 2

$$\text{Solve } 4\log_3(2x+1) - 7 = 1.$$

Solution:

This time we have a logarithmic equation. So we must isolate the logarithmic expression and exponentiate both sides. We will use a base 3 to exponentiate because $a^{\log_a x} = x$. Here is the work.

$$\begin{aligned}4\log_3(2x+1) - 7 &= 1 \\4\log_3(2x+1) &= 8 \\\log_3(2x+1) &= 2 \\3^{\log_3(2x+1)} &= 3^2 \\2x+1 &= 9 \\x &= 4\end{aligned}$$

So our solution is $x = 4$. But we need to check to make sure that this does not violate the domain of our equation. So plugging 4 into $\log_3(2x+1)$ we get $\log_3 9$, which is okay. Hence our solution is $x = 4$.

Lets proceed to more challenging questions.

Example 3

$$\text{Solve } \frac{50}{2 - 7^{4t-1}} = 780.$$

Solution:

We are again dealing with an exponential equation. So we must isolate the exponential part. We use techniques that we learned before. Lets begin by multiplying both sides of the equation by the LCD (i.e. $2 - 7^{4t-1}$). This gives

$$50 = 780(2 - 7^{4t-1})$$

Now it is just like before. We can isolate, and take the logarithm of both sides. Let's use base 7 so that we can use our properties.

$$\begin{aligned}\frac{50}{780} &= 2 - 7^{4t-1} \\7^{4t-1} &= \frac{151}{78} \\\log_7 7^{4t-1} &= \log_7 \frac{151}{78} \\4t - 1 &= \log_7 \frac{151}{78} \\t &= \frac{1}{4} \left(\log_7 \frac{151}{78} + 1 \right)\end{aligned}$$

So, our solution is $t = \frac{1}{4} \left(\log_7 \frac{151}{78} + 1 \right)$. But we must use the change of base formula to get the decimal equivalent answer. That is,

$$t = \frac{1}{4} \left(\log_7 \frac{151}{78} + 1 \right) = \frac{1}{4} \left(\frac{\ln \frac{151}{78}}{\ln 7} + 1 \right) \approx 0.3349$$

Example 4

Solve $\log_8 2x + \log_8 (x-3) = 1$.

Solution:

This is a logarithmic equation. So we must isolate the logarithm. However, there are two of the logarithms there. But they are separated by a plus sign and we know that we can condense logarithms that are separated by a plus or minus. We solve as follows

$$\begin{aligned} \log_8 2x + \log_8 (x-3) &= 1 \\ \log_8 2x(x-3) &= 1 \\ 8^{\log_8 2x(x-3)} &= 8^1 \\ 2x(x-3) &= 8 \\ 2x^2 - 6x - 8 &= 0 \\ 2(x+1)(x-4) &= 0 \\ x = -1 & \quad x = 4 \end{aligned}$$

But checking $x = -1$ in the original equation we get $\log_8 -2$ which violates the domain, but 4 checks out okay. Thus the solution is $x = 4$.

Notice in the last example we can not just exponentiate each piece of an equation. We can only exponentiate an entire side of the equation.

The common mistake is $\log_8 2x + \log_8 (x-3) = 1 \Rightarrow 8^{\log_8 2x} + 8^{\log_8 (x-3)} = 8^1$

Which gives $2x + (x-3) = 8$ or $x = \frac{11}{3}$, which is incorrect.

Example 5

Solve $\log_9 x = \log_9 (x-1) + \frac{1}{2}$.

Solution:

Again we must isolate the logarithm first. Careful to not exponentiate each piece of the equation, but only exponentiate the entire side.

$$\begin{aligned} \log_9 x &= \log_9 (x-1) + \frac{1}{2} \\ \log_9 x - \log_9 (x-1) &= \frac{1}{2} \end{aligned}$$

$$\log_9 \left(\frac{x}{x-1} \right) = \frac{1}{2}$$

$$9^{\log_9 \left(\frac{x}{x-1} \right)} = 9^{\frac{1}{2}}$$

$$\frac{x}{x-1} = 3$$

$$x = 3(x-1)$$

$$-2x = -3$$

$$x = \frac{3}{2}$$

So the solution is $x = \frac{3}{2}$. Upon checking in the original equation, the solution is okay.

11.5 Exercises

Solve the following. Give both exact and decimal approximate answers.

1. $2^x = 5$

2. $5^x = 3$

3. $\log_3 x = 4$

4. $\log_2 x = 7$

5. $7^{x+1} = 6$

6. $3^{2x-1} = 5$

7. $\ln(x+3) = 6$

8. $\ln(2-x) = 4$

9. $7 - e^{7x} = 1$

10. $\log_5(x+4) - 3 = 6$

11. $3 - \ln(x-2) = 5$

12. $4 - e^{x-1} = 3$

13. $4(0.1)^x = 8$

14. $5(0.7)^{x+1} = 25$

15. $\ln(2x+3) - 1 = 0$

16. $-5 + \log_7 6x = 0$

17. $3\ln(2x+3) = 9$

18. $5\log_2(4x-3) = 50$

19. $5(2)^{3x} - 4 = 13$

20. $-16 + 0.2(10)^x = 35$

21. $8 - 12e^{-x} = 7$

23. $4 - 2e^x = -23$

24. $\frac{8000}{(1.03)^t} = 6000$

25. $\frac{10}{(1+0.05)^{12t}} = 205$

26. $\frac{10,000}{\left(1 + \frac{0.06}{4}\right)^{4t}} = 5000$

27. $\frac{15}{\left(1 + \frac{0.055}{2}\right)^{2t}} = 5$

28. $1 - 2\ln x = -4$

29. $\ln(x+1) + 2 = 4$

30. $\frac{500}{1 + e^{-0.1t}} = 400$

31. $\frac{55}{2 - e^{-4t}} = 11$

32. $\frac{1}{5}4^{x+2} = 300$

33. $10 - \frac{2}{3}6^{3x} = 2$

34. $23 - 5e^{x+1} = 3$

35. $\frac{2}{3}\ln(x+1) = -1$

36. $\log_4(x^2 - 3x) = 1$

37. $\log 100 - \log(x+1) = -1$

38. $-5 + 3\log_5 3x = 4$

39. $\log_3 x + \log_3(x-2) = 1$

40. $\log_2(x-1) + \log_2(x+3) = 3$

41. $\log_3 x + \log_3(2x-3) = 2$

42. $\log x + \log(x-15) = 2$

43. $\ln(x+2) = \ln(x-4) + \ln 3$

44. $\frac{1}{5}4^{x+2} - 50 = 250$

45. $2000e^{-2t} = 4000$

46. $-1 + 3\log_{10} \frac{x}{2} = 8$

47. $\ln \sqrt{x} = 9$

48. $\left(1 + \frac{0.10}{12}\right)^{12t} = 2$

49. $20\left(100 - e^{x/2}\right) = 500$

50. $\log_3 x + \log_3(x^2 - 8) = \log_3 8x$

51. $\log_4 x - \log_4(x - 1) = \frac{1}{2}$

52. $\frac{3000}{2 + e^{2x}} = 2$

53. $\ln x + \ln(x - 2) = 1$

54. $\log_3 x + \log_3(x + 6) = 3$

55. $\log_3 6 - \log_3(2x + 3) = \log_3(x + 1)$

56. $20\left(100 - e^{x/2}\right) = 500$

57. $\ln x + \ln(x + 1) = 0$

58. $3^{2x} - 5 \cdot 3^x + 6 = 0$ (Hint: Use a substitution)

59. $e^{2x} - 4e^x - 5 = 0$ (Hint: Use a substitution)

60. $\ln x - \ln(x - 1) = 1$

61. $\log_6 x = 1 - \log_6(x - 1)$

62. $\log_2(x + 4) = 2 - \log_2(x + 1)$

63. $\ln(x + 1) = 2 + \ln(x - 1)$

64. $\log_{10}(x - 6) + \log_{10}(x + 3) = 1$

65. $\log_8(x + 2) + \log_8(x - 3) = \frac{1}{3}$

Find the x- and y-intercepts of the following.

66. $y = \ln(3 - x) + 7$

67. $y = 2 \log_3(x + 1)$

68. $y = 6 - e^{2-x}$

69. $y = 3e^{x+1}$

70. $y = 1 - 2 \log_2(x - 1)$

71. $y = 4 \cdot 7^x - 3$

72. $y = 5 - 4^{x+3}$

73. $y = -\log_5(2 - x) - 3$

Find the inverse of the following functions.

74. $f(x) = 2^{x+1}$

75. $f(x) = 3^x - 1$

76. $f(x) = \log_2 x - 3$

77. $f(x) = \log_3(x + 1)$

78. $f(x) = e^{x+2} - 1$

79. $f(x) = 5^{x-4} + 3$

80. $f(x) = \log_5(6x + 4) - 3$

81. $f(x) = -\ln(3x) - 4$

Solve for the indicated variable.

82. $M = \log \frac{I}{I_0}$ for I_0 .

83. $A = Pe^{rt}$ for t .

84. $A = Pe^{rt}$ for r .

85. $y = Ce^{kt}$ for k .

86. $y = Ce^{kt}$ for C .

87. $pH = -\log(H^+)$ for H^+

88. $T = T_s + D_0 e^{-kt}$ for t .

89. $T = T_s + D_0 e^{-kt}$ for k .

90. $y = \frac{a}{1 + be^{-(x-c)/d}}$ for x .

91. $y = \frac{a}{1 + be^{-(x-c)/d}}$ for d .

92. $y = ae^{-(x-b)^2/c}$ for c .

93. $y = a \log_b(x - h) + k$ for x .

