11.4 Properties of Logarithms

The next thing we want to do is talk about some of the properties that are inherent to logarithms.

Properties of Logarithms	
1. $\log_a(uv) = \log_a u + \log_a v$	1. $\ln(uv) = \ln u + \ln v$
2. $\log_a(u/v) = \log_a u - \log_a v$	2. $\ln(u/v) = \ln u - \ln v$
3. $\log_a u^n = n \log_a u$	3. $\ln u^n = n \ln u$

There are several applications to these properties that we will need later. Some of which will appear in the next section.

However, currently we just want a familiarity with the properties and we want to perform two basic types problems: expanding and condensing logarithms.

Before we get to the expanding and condensing problems, we want to first notice that the properties take the operations of multiplying, dividing or exponents from the inside of a logarithms and turn them into adding, subtracting or coefficients on the outside of the logarithm, or vice versa.

<u>Example 1</u>:

Condense the following logarithms.
a. $4 \ln 2 + 2 \ln x - \ln y$ b. $1/5 [\ln x - 2\ln (x + 4)]$ c. $\frac{1}{4} \log_x 3 - \frac{1}{2} \log_x y - \log_x z$

Solution:

By condense the log, we really mean write it as a single logarithm with coefficient of 1. So we will need to use the properties above to condense these logarithms.

a.   The first thing we must do is move the coefficients from the front into the exponents by using property 3. This gives us
$$4 \ln 2 + 2 \ln x - \ln y = \ln 2^4 + \ln x^2 - \ln y$$
Now by properties 1 and 2 we can condense the addition and subtractions into multiplication and division. We get
$$\ln 2^4 + \ln x^2 - \ln y = \ln 2^4 x^2 - \ln y$$
$$= \ln \frac{2^4 x^2}{y}$$
$$= \ln \frac{16x^2}{y}$$
So, $4 \ln 2 + 2 \ln x - \ln y = \ln \frac{16x^2}{y}$

b.   This time we need to follow the order of operations a little closer. So we must start inside the brackets and work our way out. This means that the 1/5 will be the last thing we take care of. So proceeding like part a. we condense the coefficient then the subtraction as follows
$$1/5 [\ln x - 2\ln (x + 4)] = 1/5 [\ln x - \ln (x + 4)^2]$$

$$= 1/5 \ln \frac{x}{(x+4)^2}$$
Now we can deal with the 1/5. Pulling it up as an exponent we get
$$1/5 \ln \frac{x}{(x+4)^2} = \ln \left(\frac{x}{(x+4)^2} \right)^{1/5}$$

$$= \ln \sqrt[5]{\frac{x}{(x+4)^2}} \text{ since 1/5 power is the same as a } 5^{th} \text{ root.}$$

So 1/5 [ln x – 2ln (x + 4)] = ln $\sqrt[5]{\dfrac{x}{(x+4)^2}}$

c. Finally, we proceed as we did in the two previous examples. We will turn the coefficients into exponents and then condense the subtraction. We get

$$\tfrac{1}{4}\log_x 3 - \tfrac{1}{2}\log_x y - \log_x z = \log_x 3^{1/4} - \log_x y^{1/2} - \log_x z$$

$$= \log_x \frac{3^{1/4}}{y^{1/2}} - \log_x z$$

$$= \log_x \frac{\dfrac{3^{1/4}}{y^{1/2}}}{z} \quad \text{which we need to simplify.}$$

So we recall to simplify a complex fraction we multiply numerator and denominator by the lcd. That is $y^{1/2}$ here. This gives us

$$\log_x \frac{\dfrac{3^{1/4}}{y^{1/2}}}{z} = \log_x \frac{3^{1/4}}{y^{1/2} z}$$

$$= \log_x \frac{\sqrt[4]{3}}{z\sqrt{y}}$$

So $\tfrac{1}{4}\log_x 3 - \tfrac{1}{2}\log_x y - \log_x z = \log_x \dfrac{\sqrt[4]{3}}{z\sqrt{y}}$

Example 2:

Expand the following logarithms.

a. $\ln \sqrt[3]{\dfrac{x^2}{x+1}}$ b. $\log_4[4x(x-5)]^2$ c. $\log_3 \dfrac{9}{x^2\sqrt{y}}$

Solution:

By expand the logarithm, we basically mean we want to write it out as it was given to us in example 1. That is, if there is any property that can be done two write the logarithm, then we must do it. We basically perform expanding by doing condensing backwards.

a. First, we need to change the radical into a rational exponent. $\ln \sqrt[3]{\dfrac{x^2}{x+1}} = \ln\left(\dfrac{x^2}{x+1}\right)^{1/3}$

Now we pull the exponent out to the front as a coefficient and use property 2 to turn the division inside the ln into a subtraction outside. We must be careful to leave the coefficient on the entire expression. This gives us

$$\ln\left(\frac{x^2}{x+1}\right)^{1/3} = \frac{1}{3}\ln\frac{x^2}{x+1}$$

$$= \frac{1}{3}\left[\ln x^2 - \ln(x+1)\right]$$

Finally, we can pull the 2nd power to the front of the lnx. So we have

$$\ln \sqrt[3]{\frac{x^2}{x+1}} = \frac{1}{3}\left[2\ln x - \ln(x+1)\right]$$

Notice that we cannot do any expanding to ln (x + 1) since the properties clearly state that it is addition outside not addition inside that can be changed.

b. Again we will start with pulling the exponent to the front as a coefficient, then use property 2 to expand the multiplications out.

$$\log_4[4x(x - 5)]^2 = 2\log_4[4x(x - 5)]$$
$$= 2\,(\log_4 4 + \log_4 x + \log_4 (x - 5))$$

Now we notice that we have the expression $\log_4 4$. If we evaluate this we get 1.
Therefore what we really have is

$$\log_4[4x(x - 5)]^2 = 2\,(1 + \log_4 x + \log_4 (x - 5))$$
$$= 2 + 2\log_4 x + 2\log_4 (x - 5)$$

c. Lastly, we will change the radical to a rational exponent and go about the problem in the same fashion as the previous examples.

$$\log_3 \frac{9}{x^2 \sqrt{y}} = \log_3 \frac{9}{x^2 y^{\frac{1}{2}}}$$
$$= \log_3 9 - \log_3 x^2 y^{1/2}$$
$$= \log_3 9 - [\log_3 x^2 + \log_3 y^{1/2}]$$
$$= \log_3 9 - [2\log_3 x + \tfrac{1}{2}\log_3 y]$$
$$= 2 - 2\log_3 x - \tfrac{1}{2}\log_3 y$$

11.4 Exercises

Condense.

1. $\log_2 x + \log_2 y$
2. $\log_3 y - \log_3 z$
3. $3\log_5 v$
4. $\ln u + \ln x$
5. $\ln x - \ln y$
6. $7\log_4 x$
7. $2\log_4 p + \log_4(q + 1)$
8. $\log_5 x - 4\log_5 m$
9. $\ln x - 4\ln y$
10. $5\log_3(a - 1) - \log_3 b$
11. $4\log_5 b + 2\log_5 c$
12. $3\ln(a + c) + 4\ln(c - d)$
13. $\log_2 3 + \log_2(x + 4) - \log_2 y$
14. $\log_5 4 - \log_5 a + \log_5 b$
15. $2\ln x + 3\ln y + \ln z$
16. $3\log u + 3\log v + 2\log w$
17. $\log_7 n - 4\log_7 3 + 2\log_7 t$
18. $3\log_3 y - 2\log_3 z + \log_3 x$
19. $\tfrac{1}{2}\log_2 x - 2\log_2(y + 1)$
20. $\ln(p + r) + \tfrac{1}{3}\ln(q - r)$
21. $\tfrac{1}{4}\ln(x - y) - 3\ln y$
22. $2(\log_4 x + \log_4 y) - \tfrac{1}{2}\log_4 z$
23. $\tfrac{1}{3}\log_2 n + 2(\log_2 m - 3\log_2 l)$
24. $4(\ln x + \ln y + \ln z)$
25. $\tfrac{1}{2}(\ln x + \ln y - \ln z)$
26. $4\log_6 u - \tfrac{1}{2}\log_6 v + 2\log_6 r$
27. $3\log_3 l + \tfrac{1}{3}\log_3 w - 2\log_3 h$
28. $\log_2 x - \log_2(y - 1) - \log_2(z + 3)$
29. $4\log_5(p + q) - \log_5(q - r) - \log_5 r$
30. $\tfrac{1}{2}(\ln(u+1) - \ln(v-1)) + \tfrac{1}{3}\ln w$
31. $\tfrac{1}{3}(\log_4 5 + \log_4(x + y)) - 2\log_4 7$
32. $2\log_2 x + 3\log_2 y - \tfrac{1}{2}(\log_2 a + \log_2 b)$
33. $\tfrac{1}{3}(\log_4 x + \log_4 y) - 2(\log_4 a + \log_4 b)$
34. $\tfrac{1}{2}\log_3 u + \tfrac{1}{3}(\log_3 v - \log_3 w) + \log_3 x$
35. $\tfrac{1}{3}\log_2 x - 2(\log_2 n - 4\log_2 m)$
36. $2\log_3 x - \tfrac{1}{2}(\log_3 y + 5\log_3 z)$
37. $\tfrac{1}{2}(\ln x + 3\ln y) - \tfrac{1}{3}\ln z$
38. $\tfrac{1}{3}[\ln u - 2\ln v - \ln x]$
39. $\ln(a + 1) - \tfrac{1}{3}\ln(b - 1) - 3(\ln(c + 2) - \ln(d - 2))$
40. $2\ln(w - 5) - \tfrac{1}{2}(\ln(x + y) - \ln(x - y)) - 3\ln z$

Expand.

41. $\log_2 xy$
42. $\log_3 \dfrac{x}{y}$
43. $\log_4 x^3$

44. $\log_5 nm$

45. $\ln \dfrac{x}{y}$

46. $\ln x^7$

47. $\log_5 x^2 y$

48. $\log_2 yz^2$

49. $\ln \dfrac{x}{y^2}$

50. $\ln \dfrac{u^2}{v^3}$

51. $\log_3 \dfrac{uv^2}{w}$

52. $\log_4 \dfrac{x^3 y}{z^2}$

53. $\log_7 \dfrac{x}{yz}$

54. $\log_6 \dfrac{x^2}{pq^2}$

55. $\log_2 \sqrt{x+y}$

56. $\log_3 \sqrt{xy}$

57. $\ln x\sqrt{y}$

58. $\ln \dfrac{\sqrt{y}}{u^2}$

59. $\log_3 \dfrac{u\sqrt{v}}{w^2}$

60. $\log_4 \dfrac{n}{\sqrt{pq}}$

61. $\log_4 \dfrac{\sqrt[3]{xy}}{z}$

62. $\log_3 \dfrac{n}{m\sqrt[3]{r}}$

63. $\ln \sqrt[3]{\dfrac{xy}{ab}}$

64. $\log_2 2x(x-1)$

65. $\log_3 \dfrac{3}{\sqrt{x+1}}$

66. $\log_5 \dfrac{\sqrt{5}}{xy}$

67. $\log_4 16u(u+v)$

68. $\log_3 \dfrac{27x^2}{\sqrt{y}}$

69. $\log_2 \dfrac{x+1}{\sqrt[3]{2}}$

70. $\ln e^2 x$

71. $\ln \sqrt{e(x+1)}$

72. $\log_7 \sqrt{\dfrac{49}{a+b}}$

73. $\log_6 216(x+h)^2$

Factor the inside of the logarithm. Then expand.

74. $\log_4 (x^2 -1)$

75. $\ln(x^2 - 2x + 1)$

76. $\log_2(4x^2 - 12x + 8)$

77. $\ln(ex^2 - 4e)$

78. $\log_2 \dfrac{32x}{x^2 -1}$

79. $\log_2 \dfrac{x^2 - 2x + 1}{16}$

80. $\ln \sqrt{\dfrac{x^2 - 2x + 1}{x^2 + 2x + 1}}$

For Question 81 – 85 use the fact that $\log A = a$, $\log B = b$, and $\log C = c$ to evaluate the logarithmic expression in terms of a, b, and c.

81. $\log AB$

82. $\log B^2$

83. $\log A^3 C$

84. $\log\left(\frac{AB}{C^2}\right)$

85. $\log\left(\frac{\sqrt{A}}{B^3 C}\right)$

For questions 86 – 90 use the fact that $\log_b 5 \approx 1.61$, $\log_b 3 \approx 0.792$, and $\log_b 7 \approx 1.404$ to evaluate the following logarithmic expressions.

86. $\log_b 15$

87. $\log_b 49$

88. $\log_b 105$

89. $\log_b\left(\frac{3}{5}\right)$

90. $\log_b\left(\frac{49}{125}\right)$