11.2 Logarithmic Functions

In the last section we dealt with the exponential function. One thing that we notice from that discussion is that all exponential functions pass the horizontal line test. That means that the exponential function must have an inverse function. If we try to find this inverse function using our standard methods we would get the following:

$$f(x) = a^x$$
$$y = a^x$$
$$x = a^y$$

We see that we are stuck. For this reason we simply define the inverse of an exponential function in such a way that it satisfies the properties of inverse functions. We call this function the logarithmic function with base *a*.

Definition: Logarithmic function-
Let *a* and *x* be positive real numbers such that $a \neq 1$. The logarithm of x with base a is written $\log_a x$ and is defined as follows: $$y = \log_a x \text{ if and only if } x = a^y$$ $f(x) = \log_a x$ is the logarithmic function with base *a*.

For example, the following is a true statement $\log_3 \frac{1}{243} = -5$. The way we can verify this is by using the above definition to change the statement into exponential form (since we can deal easily with exponentials). If we change it over we get $3^{-5} = \frac{1}{243}$ which is clearly a true statement.

There are two primary techniques we use to change equations from logarithmic to exponential form. The first is to just use the definition above by labeling each of the values *x, y,* and *a*. Then just substituting them into the definition in the proper spot.

The other way of changing between logarithmic and exponential forms is by using a much more descriptive statement to decipher what each piece of a logarithmic expression is. That statement is the following: **log $_{\text{base}}$ answer = exponent**, where "base" is the base of the exponential, "answer" is the opposite side of the equal sign of the exponential, and "exponent" is the exponent of the exponential. Many people prefer this method because it is much more descriptive than trying to remember two expressions that both contain *x, y,* and *a.*

Example 1:

Evaluate the following.

a. $\log_3 27$ b. $\log_5 1$ c. $\log_2 0$ d. $\log_5 5^3$

Solution:

a. Since we want to know the value of $\log_3 27$, lets say that its x. Then we have

$$\log_3 27 = x$$

Now using **log $_{\text{base}}$ answer = exponent** we can change the expression over to exponential form. We can see that the base is 3, answer is 27 and exponent is x. This gives us

$$3^x = 27$$

Clearly, x = 3 since $3^3 = 27$. So $\log_3 27 = 3$.

b. Similarly, we will set $\log_5 1 = x$ and find x by using **log** $_{\textbf{base}}$ **answer = exponent** to convert to exponential form. This gives us $5^x = 1$. So $x = 0$ since $5^0 = 1$.
So $\log_5 1 = 0$.

c. Again set $\log_2 0 = x$ and convert to exponential form. This gives us $2^x = 0$. But no such x will exists since any power of 2 is greater than zero. There for we say $\log_2 0$ is not a real number. This also tells us that 0 is not in the domain of $y = \log_2 x$ (a fact that we will need later).

d. Lastly, set $\log_5 5^3 = x$ and convert to exponential form. We get $5^x = 5^3$. Clearly $x = 3$. So therefore $\log_5 5^3 = 3$.

<u>Example 2</u>:

Solve the following for x.

a. $\log_5 x = 1$ b. $\log_4 x = 0$

Solution:

a. Again using **log** $_{\textbf{base}}$ **answer = exponent** we will convert the equation to exponential form. So $\log_5 x = 1$ becomes $5^1 = x$. So clearly $x = 5$.

b. Likewise we convert $\log_4 x = 0$ to $4^0 = x$. So $x = 1$.

These previous examples give us the following properties.

Basic Properties of Logarithms	
1. $\log_a 1 = 0$	because $a^0 = 1$
2. $\log_a a = 1$	because $a^1 = a$
3. $\log_a a^x = x$	because $a^x = a^x$

We will need these properties later.

Next we want to talk about two logarithms that have special bases. First,

<u>The Common Log</u>
$\log_{10}$ is called the <u>common log</u>. We usually write just log to mean the common log. Also, common log is on every scientific calculator. It's the button $\boxed{\log}$

<u>Example 3</u>:

Use your calculator to evaluate log 9. Round to three decimal places.

Solution:

Using your calculator input log of 9. Depending on the type you have you might have to input log first or 9 first. Consult your instructions or ask you instructor for help. You should get
$\log 9 = 0.9542425\ldots$
So rounding off we get $\log 9 = 0.954$.

<u>The Natural Log</u>
Recall from the last section we defined the natural base, *e*. Therefore, the function defined by $f(x) = \log_e x = \ln x$ is called the <u>natural logarithmic function</u>. This is also on your calculator. It's the button $\boxed{\ln}$

So ln x is just a logarithm with a base of *e*. We never write $\log_e$. It is considered improper notation.

Example 4:

Use your calculator to evaluate ln 6. Round to three decimal places.

Solution:

Using your calculator input ln of 6. Again depending on the type you have you might have to input ln first or 6 first. Consult your instructions or ask you instructor for help. You should get
ln 6 = 1.791759…
So rounding off we get ln 6 = 1.792.

Example 5:

Solve the following for x. Round to three decimal places.

a. log x = -2.1 b. ln x = 1.4

Solution:

Since this time we are not looking for the value of the logarithm we cannot do the problem like examples 3 or 4. So, since this time we have the output and we want to know what will give each of these values we just use the shift, 2^{nd}, or inv buttons on our calculator.

a. So since we have the output of –2.1 we hit the 2^{nd} button before the log button. Consult your instructions or ask you instructor for help. We should get x = 0.008 (with rounding).

b. Similarly we use the 2^{nd} button before the ln button. We should get x = 4.055 (with rounding).

Lets say we wanted to evaluate $\log_5 7$. The problem is if we change it to exponential form we get $5^x = 7$, which does not seem easy, and we don't have a $\log_5$ button on our calculator to solve it. In order to rectify that situation we have a formula for changing from any base log to another base that we can use (usually base 10 or e). This formula is called the change of base formula.

Change of Base Formula
$\log_a x = \frac{\log_b x}{\log_b a}$ or specifically $\log_a x = \frac{\ln x}{\ln a}$

We can use this to evaluate any log with any base.

Example 6:

Evaluate. Round your answer to three decimal places.

a. $\log_5 7$ b. $\log_5 510$ c. $\log_3 (1+e^3)$ d. $\log_{1/3} 0.015$

Solution:

a. We use the change of base formula to change from a base 5 to any base we want. Since we want to use our calculators to find the value, we will use either base e or base 10. Lets use base e, that is ln. So we have

$$\log_5 7 = \frac{\ln 7}{\ln 5}$$
$$\approx 1.209$$

b. Again we will use the change of base formula. This time lets change to base 10. This gives

$$\log_5 510 = \frac{\log 510}{\log 5}$$
$$\approx 3.874$$

c. Since it doesn't matter what base we choose for the change of base formula we will use ln for simplicity. This gives

$$\log_3\left(1+e^3\right) = \frac{\ln\left(1+e^3\right)}{\ln 3}$$
$$\approx 2.775$$

d. Finally, change of base formula gives us

$$\log_{1/3} 0.015 = \frac{\ln 0.015}{\ln \frac{1}{3}}$$
$$\approx 3.823$$

11.2 Exercises

Evaluate without using a calculator.

1. $\log_2 4$ 2. $\log_5 25$ 3. $\log_3 3$ 4. $\log_4 64$ 5. $\log_2 8$ 6. $\log_3 1$
7. $\log_5 125$ 8. $\log_6 216$ 9. $\log_9 1$ 10. $\log_4 4$ 11. $\log_8 64$ 12. $\log_7 49$
13. $\ln e^2$ 14. $\log 100$ 15. $\log 1000$ 16. $\ln e$ 17. $\log_7 7^3$ 18. $\log_4 4^6$
19. $\log_6 6^{20}$ 20. $\log_3 3^{43}$ 21. $\log_9 3$ 22. $\log_4 2$ 23. $\log_8 2$ 24. $\log_{27} 3$
25. $\ln \sqrt{e}$ 26. $\log_2 \sqrt{2}$ 27. $\log_{64} 4$ 28. $\log_{64} 2$ 29. $\log_{19} 1$ 30. $\log_{81} 9$
31. $\log_{81} 3$ 32. $\ln e^{1/3}$ 33. $\ln 1$ 34. $\log_y y^2$ 35. $\log_x x^3$ 36. $\ln e^x$
37. $\log_3 9 + \log_3 27$ 38. $\log_4 2 – \log_8 2$ 39. $\log_2 1 – \log_{81} 3$
40. $\log_4 2 – \log_8 2 + \log_{16} 2 + \log_{32} 2$ 41. $\log_3 3 - \log_9 3 + \log_{27} 3 – \log_{81} 3$

Solve the following for x without using a calculator.

42. $\log_4 x = 2$ 43. $\log_2 x = 3$ 44. $\log_3 x = 1$ 45. $\log_7 x = 0$
46. $\log_3 x = 3$ 47. $\log_4 x = 3$ 48. $\log_5 x = 2$ 49. $\log_6 x = 3$
50. $\log_4 x = 1/2$ 51. $\log_9 x = 1/2$ 52. $\log x = 4$ 53. $\log_8 x = 1/3$
54. $\log_{1/2} x = 2$ 55. $\log_{1/3} x = 3$ 56. $\log_{2/3} x = 1$ 57. $\log_{1/4} x = 0$
58. $\log_4 x = -2$ 59. $\log_2 x = -3$ 60. $\log_3 x = -1$ 61. $\log_6 x = -2$
62. $\log_{1/2} x = -1$ 63. $\log_{2/3} x = -2$ 64. $\log_{1/3} x = -3$ 65. $\log_{1/5} x = -3$

Evaluate. Round your answer to three decimal places.

66. $\log 5$ 67. $\ln 3$ 68. $\ln 9$ 69. $\log 13$ 70. $\log 0.73$ 71. $\log 0.0023$
72. $\ln 13.4$ 73. $\ln 3.4$ 74. $\log 33.1$ 75. $\log 0.12$ 76. $\ln 0.19$ 77. $\log 45$
78. $\ln 1.0001$

Solve the following for x. Round your answer to three decimal places.

79. $\log x = 6$ 80. $\log x = 14$ 81. $\ln x = 6$ 82. $\ln x = 8.6$
83. $\log x = -56$ 84. $\ln x = -16.1$ 85. $\ln x = -0.36$ 86. $\log x = -0.084$

Evaluate. Round your answer to three decimal places.

87. $\log_7 5$ 88. $\log_6 12$ 89. $\log_3 47$ 90. $\log_9 2$ 91. $\log_{13} 7$ 92. $\log_2 27$
93. $\log_{12} 22$ 94. $\log_{75} 2$ 95. $\log_7 643$ 96. $\log_{91} 42$ 97. $\log_9 36$ 98. $\log_{11} 100$
99. $\log_5 0.05$ 100. $\log_7 0.12$ 101. $\log_{3.1} 4.7$ 102. $\log_{0.12} 2.8$ 103. $\log_{1.3} 1.7$ 104. $\log_{2.5} 2.47$
105. $\log_{1/2} 6$ 106. $\log_{1/5} 26$ 107. $\log_{2/3} 1/7$ 108. $\log_{3/7} 4/5$ 109. $\log_{1/3} e$ 110. $\log_{6/13} 10$
111. $\log_{1.00034} (6.837)$ 112. $\log_{1.00002978} (2.234)$ 113. $\log_{1.0009957} (3.6573)$
114. $\log_4 (1+e^4)$ 115. $\log_6 (1/3+e)$ 116. $\log_7 (3-e^2)$

117. $\log_3 (\ln 3 + 6)$ 118. $\log_5 (\log 15 - e)$ 119. $\log(\ln(\log_4 15)$

Let $f(x) = \log_2 x$, $g(x) = \log_3 (x-1)$ and $h(x) = \ln (x+3)$. Find the following.

120. f(1) 121. f(4) 122. g(4) 123. g(2) 124. h(0) 125. h(e-3)

126. f(a) 127. g(a-b) 128. h(x+1) 129. f(x+h) 130. g(x+h) 131. h(x+h)

132. (f o g)(x) 133. (g o f)(x)