11.1 Exponential Functions

In this chapter we want to look at a specific type of function that has many very useful applications, the exponential function.

Definition: <u>Exponential Function</u>
An exponential function is a function of the form $f(x)=a^x$ where $a>0$ and $a\neq 1$, and x is any real number. a is called the <u>base</u>.

An example of an exponential function would be $f(x)=2^x$. Notice that the variable is inside the exponent. This is very different from any other function we have dealt with.

First let us recall some of the properties of exponents.

1. $a^x \cdot a^y = a^{x+y}$ 2. $\left(a^x\right)^y = a^{xy}$ 3. $\dfrac{a^x}{a^y} = a^{x-y}$ 4. $a^{-x} = \dfrac{1}{a^x} = \left(\dfrac{1}{a}\right)^x$

Some of these properties will prove to be useful as we proceed through this chapter.

Let us now do some basic evaluating of exponential functions.

<u>Example 1</u>:

Evaluate $F(x)=3^{-x}$ at $F(-2)$, $F(0)$ and $g(x)=2^{-x^2}$ at $g(3)$ and $g(-1)$.

Solution:

By the techniques we learned in chapter 9 we know

$$\begin{aligned} F(-2) &= 3^{-(-2)} \\ &= 3^2 \\ &= 9 \end{aligned} \qquad \text{and} \qquad \begin{aligned} F(0) &= 3^{-0} \\ &= 3^0 \\ &= 1 \end{aligned}$$

also,

$$\begin{aligned} g(3) &= 2^{-(3)^2} \\ &= 2^{-9} \\ &= \tfrac{1}{2^9} \\ &= \tfrac{1}{512} \end{aligned} \qquad \text{and} \qquad \begin{aligned} g(-1) &= 2^{-(-1)^2} \\ &= 2^{-1} \\ &= \tfrac{1}{2} \end{aligned}$$

A base that comes in handy in many application problems is the following

Definition: <u>The natural base</u>
$e \approx 2.71828...$ is called the natural base. Consequently, $f(x)=e^x$ is called the natural exponential function.

The number *e* is an irrational number like π which we represent with a letter for simplicity. To learn about the derivation of the number *e* take a calculus course.

We use the letter *e* to represent this value because of the mathematician Leonard Euler, who is largely responsible for its modern usage.

Lets do some problems with the natural base.

<u>Example 2</u>:

Let $f(x) = e^{-2x} + 1$. Find $f(-1)$ and $f(3)$.

Solution:

First, *e* is such a special value it can be found on your calculator. It is usually found above the button that says $\boxed{\ln}$

So

$$\begin{aligned} f(-1) &= e^{-2(-1)} + 1 \\ &= e^{2} + 1 \end{aligned}$$

Now consult your calculators instruction manual or speak with your instructor to learn how to properly input this value into your calculator. We get

$$f(-1) = e^{2} + 1 \approx 8.389$$

Similarly,

$$\begin{aligned} f(-1) &= e^{-2(3)} + 1 \\ &= e^{-6} + 1 \\ &\approx 1.00248 \end{aligned}$$

Next we want to talk about the graphs of exponential functions.

Lets do this by way of example.

<u>Example 3</u>:

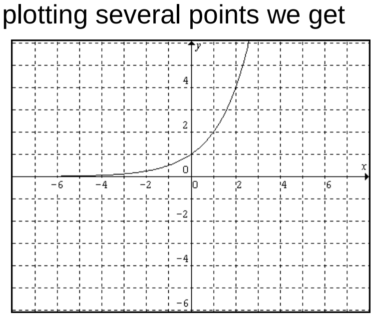
Graph the following on the same set of axis.

$f(x) = 2^{x}$ $\qquad g(x) = 4^{x}$ $\qquad h(x) = 2^{-x}$

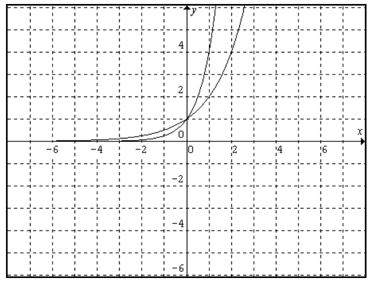
Solution:

Assuming we have no idea what the graphs of these three functions look like, we have to go back to the most fundamental way of graphing, by plotting points.

Lets first graph $f(x) = 2^{x}$. By plotting several points we get

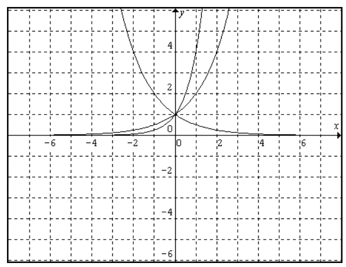


Next we graph $g(x) = 4^x$ on the same axis. We get



Notice that $g(x) = 4^x$ is rises much faster than $f(x) = 2^x$. Also notice that neither of the graphs will cross the x axis since the values for f and g could never be negative.

Lastly we want to graph $h(x) = 2^{-x}$. Intuitively, by what we learned in chapter 9, this should be the graph of f reflected about the y-axis. Plotting points will verify this fact. We have



This last example tells us that everything we learned before about shifting, reflecting and dilating works the same on exponential functions as it did on the basic six functions we did before.

Summary of Graphs of Exponential functions		
	$y = a^x$	$y = a^{-x}$
Domain:	$(-\infty, \infty)$	$(-\infty, \infty)$
Range:	$(0, \infty)$	$(0, \infty)$
Intercept:	$(0,1)$	$(0,1)$
All the same rules for shifting and reflecting are still valid.		

So we will graph exponential functions in a similar fashion to how we graphed basic function.

<u>Example 4</u>:

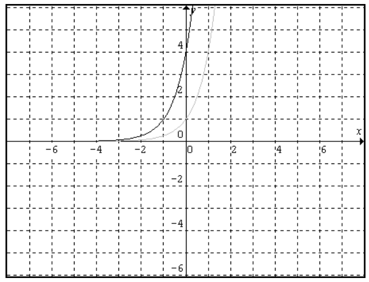
Graph the following.

a. $f(x) = 4^{x+1}$ b. $g(x) = -5^{x-2} - 2$

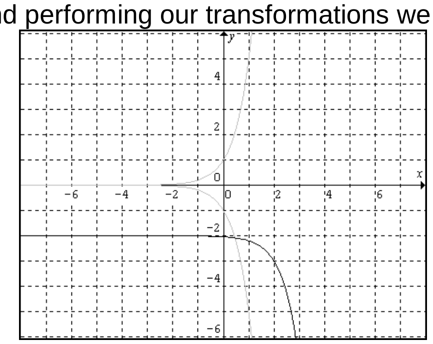
Solution:

a. From the previous example we know that the shape of the graph of an exponential function is slightly different depending on the base. Therefore, we will have to plot a few points of the basic function to account for this variation.

Here we clearly have the basic function $y = 4^x$ with a shift of 1 unit left. So by graphing the basic function (by plotting 3 points) and then shifting it we have



b. Similarly we have the basic function $y = 5^x$ with a reflection across the x-axis, and shifting 2 units down and 2 right. Recall we want to perform the reflections first. So plotting 3 points and performing our transformations we have



Finally, we want to introduce one of the many applications inherent to exponential functions: Compound interest.

Essentially, compounding ones interest means getting interest on your interest. That is, if you get your interest 2 times a year, you first get interest on the amount in the account, say \$100 gives you interest of \$1. Then the second compounding you now get interest on \$101. Which would be more than the interest on \$100. Here are the formulas for compounded interest.

Compound Interest
The amount of money A after time t, in an account with interest rate r, and principle P is given by the following: For n compoundings per year: $A = P\left(1+\frac{r}{n}\right)^{nt}$ For continuous compounding: $A = Pe^{rt}$

Continuous compounding means that you get your interest compounded on a continuous basis, i.e. an infinite amount of times per year.

<u>Example 5</u>:

Jon and Matt each have \$10,000 they received from doing some extra work at home. Jon invests his money in an account that pays him 5.3% and compounds his money daily. Matt invests his money in an account that pays him 5.2% and compounds his money continuously. Who has more money at the end of 5 years?

Solution:

Lets treat each of them separately to determine how much they have at the end of 5 years.

Jon:
First identify which type of interest problem this is. Since it says compounding daily we will need to use the formula $A = P\left(1+\frac{r}{n}\right)^{nt}$. Next identify each of the values involved.
$P = \$10{,}000$, $r = 0.053$, $n = 365$ and $t = 5$. Putting these into the formula we have

$$\begin{aligned} A &= P\left(1+\frac{r}{n}\right)^{nt} \\ &= 10{,}000\left(1+\frac{0.053}{365}\right)^{3655} \\ &= 10{,}000(1.0001452)^{1825} \\ &= \$13{,}033.93 \end{aligned}$$

Matt:
Similarly we see since this is a continuously compounding account we need the formula $A = Pe^{rt}$. Here $P = \$10{,}000$, $r = 0.052$ and $t = 5$. Putting these in we get

$$\begin{aligned} A &= Pe^{rt} \\ &= 10{,}000e^{0.0525} \\ &= \$12{,}969.30 \end{aligned}$$

So clearly at the end of the 5 years, Jon has more money than Matt. So the answer is Jon.

11.1 Exercises

Evaluate the given function at the given values.

1. $f(x) = 2^x$
   a. $f(2)$
   b. $f(-1)$
   c. $f(0)$
   d. $f(-4)$

2. $g(x) = 3^x$
   a. $g(3)$
   b. $g(-2)$
   c. $g(0)$
   d. $g(1)$

3. $h(x) = 4^{x-1}$
   a. $h(2)$
   b. $h(0)$
   c. $h(-1)$
   d. $h(1)$

4. $f(x) = 7^{-x}$
   a. $f(-2)$
   b. $f(2)$
   c. $f(0)$
   d. $f(-1)$

5. $g(x) = -5^x - 1$
   a. $g(1)$
   b. $g(2)$
   c. $g(0)$
   d. $g(-2)$

6. $h(x) = 3^{-|x+1|} - 2$
   a. $h(2)$
   b. $h(-2)$
   c. $h(-1)$
   d. $h(4)$

Evaluate the given function at the given values. Round your answers to three decimal places.

7. $f(x) = e^x$
   a. $f(2)$
   b. $f(1)$
   c. $f(0)$
   d. $f(-1)$

8. $g(x) = -e^x$
   a. $g(0)$
   b. $g(-2)$
   c. $g(-1)$
   d. $g(2)$

9. $h(x) = e^{1-x} - 1$
   a. $h(0)$
   b. $h(-3)$
   c. $h(2)$
   d. $h(-4)$

10. $f(x) = e^{2x} - 2$
   a. $f(1)$
   b. $f(\frac{1}{2})$
   c. $f(-1)$
   d. $f(0)$

11. $g(x) = -e^{x^2-1}$
   a. $g(0)$
   b. $g(1)$
   c. $g(-1)$
   d. $g(2)$

12. $h(x) = -6^x - 4$
   a. $h(0.1)$
   b. $h(\frac{1}{3})$
   c. $h(1.8)$
   d. $h(-0.7)$

13. $f(x) = 10^{x-1}$
   a. $f(1)$
   b. $f(1.1)$
   c. $f(-0.3)$
   d. $f(3.5)$

14. $g(x) = 5^{-x} - 2$
   a. $g(0.25)$
   b. $g(-0.75)$
   c. $g(2.3)$
   d. $g(-1.68)$

Let $f(x) = 2^{x+1}$, $g(x) = 5^{2x-1}$ and $h(x) = x^2 + 7x$. Find the following.

15. $f(t)$
16. $g(a+1)$
17. $f(x-1)$
18. $f(x+a)$
19. $g(x+a)$
20. $(f \circ h)(x)$
21. $(g \circ h)(x)$
22. $(h \circ g)(x)$
23. $(h \circ f)(x)$
24. $(f \circ g)(x)$
25. $(g \circ f)(x)$
26. $(f \circ f)(x)$

Graph the following.

27. $f(x) = 5^{-x}$
28. $g(x) = -3^x$
29. $h(x) = 4^{x-2}$
30. $f(x) = 2^{x+1}$
31. $h(x) = 4^x - 1$
32. $g(x) = 5^{-x} - 2$
33. $g(x) = 3^{x-2} - 3$
34. $h(x) = -6^x - 4$
35. $f(x) = -7^{-x} + 3$
36. $g(x) = 4^{x+1} + 2$
37. $g(t) = -2^{t+3} + 4$
38. $f(x) = -2^{-(x-3)}$
39. $f(x) = 4^{-(x+1)} - 2$
40. $g(t) = -7^{t+2} - 1$
41. $h(x) = 3^{-x} + 1$
42. $h(x) = 2 - 3^{x-1}$
43. $f(x) = \left(\frac{3}{4}\right)^{x+3}$
44. $f(x) = -4^{-(x+2)}$
45. $g(t) = -6^{t-4} + 6$
46. $f(x) = -3^{-(x+3)} + 3$
47 $f(x) = -5^{-(x-3)} - 5$
48. $f(x) = e^x$
49. $g(x) = -e^x$
50. $h(x) = e^{x-1} - 1$

51. Jennifer invests \$1500 into a college fund for her son. The bank gives her an annual interest rate of 5.2% and compounds it monthly. How much money will Jennifer have for her son when he is ready for college in 15 years?

52. John invests \$7,000 in a bank account that compounds his money daily at an interest rate of 4.9%. What amount of money will John have at the end of 5 years?

53. After winning the lottery, Patrick wants to invest his money in some money market accounts. The bank offers Patrick one account, which compounds quarterly at 5.8%, and one that compounds daily at 5.6%. Which account should Patrick invest in if he starts with \$7,000,000?

54. While reading the Sunday paper, Charles notices two different banks offering different types of saving accounts. Bank of the Sequoias advertises an account with a 6.2% interest rate compounded daily and Bank of the Redwoods advertises an account with a 6.1% interest rate compounded continuously. Into which bank should Charles invest his life savings of \$30,000 if he is going to withdraw the money in 10 years for retirement?

55. Jordan and Ethan are the best of friends. They are also fierce competitors. With this in mind they each want to invest \$500. Jordan is going to invest his money into an account which pays him 2.5% interest and compounds it semiannually and Ethan is going to invest in an account which pays him 2.2% and compounds in continuously. At the end of 1 year, who has bragging rights?

56. Tracy just received a \$6,000 bonus for making a great sale. She wants to invest her money wisely. The bank offers Tracy one account, which compounds daily at 4.8%, and one that compounds continuously at 4.7%. Which account should Tracy invest in?

57. In problem 55 we met Jordan and Ethan. If they worked together and combined their money into one account when they started, they would have received more total interest. Which account would produce the most amount of interest and how much more would it be?

58. Eric wants to be a millionaire. Instead of going on a game show or going to Vegas, he decides to invest in Certificate of Deposits at his local bank. So if Eric has \$75,000 to invest in an account that compounds annually. What rate would Eric have to get in order to have one million dollars at the end of 2 years?

59. Jason needs money for college. He figures that he can make it though public school with \$30,000. Jason's local bank offers him a savings account that compounds annually. If Jason starts with \$10,000, what rate would be needed for Jason to have his college money in 2 years?

60. Eric still wants to be a millionaire. But realizing the impossibility of his situation in problem 58, he has come up with a new plan. He decides to invest in a mutual fund that guarantees him an interest rate of 7.2% compounded bimonthly. How much would Eric have to invest to have one million dollars at the end of 10 years?

61. Jared wants to save some money for a rainy day. He figures that if he should lose his pay for 2 months \$8,000 should be enough to make it through. If Jared invests his money for 5 years into an account that compounds daily at an interest rate of 7.3%, how much money would Jared have to invest?

62. A certain type of bacteria increases according to the model $P(t) = 200e^{0.236t}$ where *t* is the time in hours. How much bacteria will be present in 4 hours? 10 hours?

63. A certain type of fungus increases according to the model $P(t) = 1600e^{0.001378t}$ where *t* is the time in days. How much bacteria will be present in 7 days? 30 days?

64. The population of a small town increases according to the model $P(t) = 2000e^{0.00978t}$ where t is the time in years. How many people are in the town initially? How many people will be in the town after 20 years? 50 years?

65. The population of spiders in an abandon house increases according to the model $P(t) = 20e^{0.0251t}$ where t is the time in weeks. How many spiders were in the house initially? How many spiders will be in the house after 3 weeks? 1 year?

66. A radioactive material decays according to the model $Q(t) = 30\left(\frac{1}{2}\right)^{t/5730}$ where t is in years and Q is in grams. How much material is present initially? How much will be left after 500 years?

67. A radioactive material decays according to the model $Q(t) = 55\left(\frac{1}{2}\right)^{t/1620}$ where t is in years and Q is in grams. How much material is present initially? How much will be left after 10,000 years?