10.7 Polynomial and Rational Inequalities

In this section we want to turn our attention to solving polynomial and rational inequalities. That is, we want to solve inequalities like $x^2 - 5x + 4 < 0$.

In order to do this it would be helpful to know when the polynomial is positive and negative. This will be helpful since, as in the inequality $x^2 - 5x + 4 < 0$, we want all x values that make the polynomial less than zero, that is, negative.

We use a sign chart to find when a polynomial is positive and negative.

Creating a Sign Chart

1. Find and plot on a number line, all values where the polynomial is zero. These values are called the critical numbers.
2. Use the number line from step 1 to generate your intervals to be tested. We generally go from left to right across the number line, from one critical number to another to generate these intervals.
3. Test any value in the intervals from step 2. The sign of this value will characterize the sign of the entire interval.
4. Complete the sign chart by placing the sign of each interval directly above the interval on the graphed number line.

The reason the intervals can be generalized in this manor is that whenever the right hand side of the inequality is zero, the critical numbers cut off intervals over which the polynomial is entirely positive and negative. We will show this graphically later.

Once we have the sign chart, we simply need to look back at the original inequality to see if we wanted positive or negative values for our polynomial (determined by the < or > symbol) to complete the solution.

Example 1:

Solve and graph the solution set.

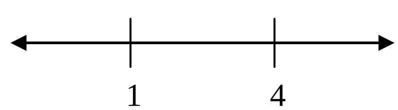
a. $x^2 - 5x + 4 < 0$ b. $(x-1)(x+5)(x-2) \geq 0$ c. $x^4 > 21x^2 - 4x^3$

Solution:

a. First we must get the polynomial into factored form, thus we can get the critical numbers. So factoring we get

$$x^2 - 5x + 4 < 0$$
$$(x-4)(x-1) < 0$$

It is easy to see that our critical numbers are $x = 1, 4$. So lets place them on the number line as follows.



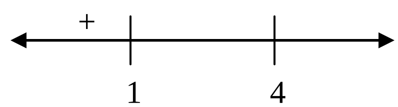
To find our intervals to be tested we must start at $-\infty$ and go from one critical number to the next until we end up going to $+\infty$. This gives us the intervals $(-\infty, 1), (1, 4), (4, \infty)$.

Now we need to test each of the intervals to see if it is positive or negative. By construction, any point in the interval will characterize the entire interval. So we use any values that we want.

For the first interval lets test zero.
To test zero, we need to put zero into the polynomial and since we are only concerned about the sign, we will not concern ourselves with the actual value at zero. It is generally easier to test the value in the factored form of the polynomial. We get

$$\begin{aligned}(0-4)(0-1) &= (-4)(-1)\\ &= -\cdot-\\ &= +\end{aligned}$$

So all values in the interval $(-\infty, 1)$ are positive. So in our sign chart, above the interval $(-\infty, 1)$ we put a + sign as follows.



We continue testing the intervals in a similar manor. For the interval $(1, 4)$ we will test the value 2 and in the interval $(4, \infty)$ we will test the value 5. We get

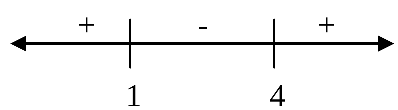
Test 2:

$$\begin{aligned}(2-4)(2-1) &= (-2)(+1)\\ &= -\cdot+\\ &= -\end{aligned}$$

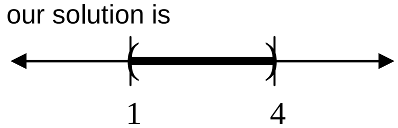
Test 5:

$$\begin{aligned}(5-4)(5-1) &= (+1)(+4)\\ &= +\cdot+\\ &= +\end{aligned}$$

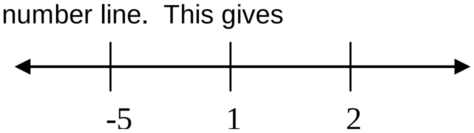
Now placing these values on our sign chart we get



Now, looking back at the original inequality we wanted to know $x^2-5x+4<0$. That is we want to know when x^2-5x+4 is negative (since the only values that are less than zero are negative values). From the sign chart we see that is represented by the interval $(1, 4)$. So the graph of our solution is



b. First of all, notice that the function is already in factored form. Thus we can read the critical number right off. We have $x=-5, 1, 2$. So now we start our sign chart. We put these critical numbers on the number line. This gives



We can see our test intervals are $(-\infty, -5), (-5, 1), (1, 2), (2, \infty)$. Now we simply test a value out of each interval.

Test –6:

$$\begin{aligned}&(-6-1)(-6+5)(-6-2)\\ &= -\cdot-\cdot-\\ &= -\end{aligned}$$

Test 0:

$$\begin{aligned}&(0-1)(0+5)(0-2)\\ &= -\cdot+\cdot-\\ &= +\end{aligned}$$

Test 1.5:

$(1.5-1)(1.5+5)(1.5-2)$

$=+\cdot+\cdot-$

$=-$

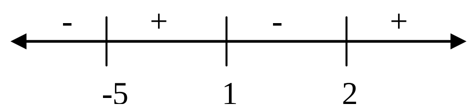
Test 3:

$(3-1)(3+5)(3-2)$

$=+\cdot+\cdot+$

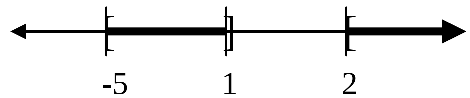
$=+$

Putting this on the sign chart we get



Since our inequality is $(x-1)(x+5)(x-2)\geq 0$ and the only values greater than zero are positives, we take the positive parts of our sign chart as the solution. We must also include the endpoints of these intervals since the inequality is a greater that **or equal to** symbol. The endpoints are the values that are equal to zero. So we have the solution is

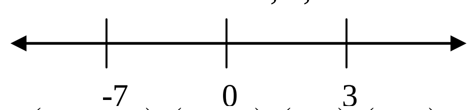
$$[-5,\ 1]\cup[2,\ \infty)$$



c. This time we need to start by getting a zero on the right side and factoring completely. We have

$$x^4 > 21x^2 - 4x^3$$
$$x^4 + 4x^3 - 21x^2 > 0$$
$$x^2\left(x^2 + 4x - 21\right) > 0$$
$$x^2(x+7)(x-3) > 0$$

So clearly our critical numbers are $x=-7,\ 0,\ 3$. So we start our sign chart



Our test intervals are $(-\infty,\ -7),\ (-7,\ 0),\ (0,\ 3),\ (3,\ \infty)$. So we test as follows

Test –8:

$(-8)^2(-8+7)(-8-3)$

$=+\cdot-\cdot-$

$=+$

Test –1:

$(-1)^2(-1+7)(-1-3)$

$=+\cdot+\cdot-$

$=-$

Test 1:

$1^2(1+7)(1-3)$

$=+\cdot+\cdot-$

$=-$

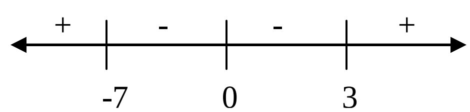
Test 4:

$4^2(4+7)(4-3)$

$=+\cdot+\cdot+$

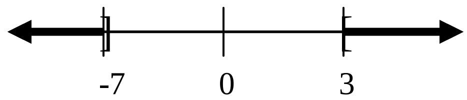
$=+$

So our sign chart is



Since the inequality is $x^2(x+7)(x-3)>0$ we want the positive values. Thus our solution is

$$\left(-\infty,\,-7\right)\cup\left(3,\,\infty\right)$$



As we mentioned at the outset, we want to not only solve polynomial inequalities, but also rational inequalities as well. The only difference in these types of inequalities is the values you get for the critical numbers.

Critical Numbers for Rational Inequalities
The critical numbers for a rational inequality are all values where the expression is zero and undefined. That is, where the numerator and denominator are zero.

There are a few things to be aware of however, when dealing with rational inequalities.

First, as with polynomial inequalities, we need to have a zero on the right hand side. Also, to use the above rule for critical numbers, we need to have one single fraction. That mean we need to combine all of the terms over an LCD. The last thing to be aware of is that when dealing with rational expressions, the domain is always an issue. So that means that the critical numbers that came from which the denominator can never be included in the solution set. We will see just how to deal with this in our next example.

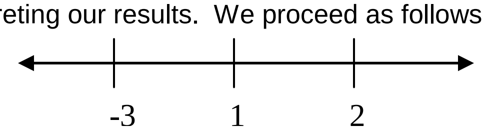
<u>Example 2</u>:

Solve and graph the following.

a. $\dfrac{(x+3)(x-1)}{x-2}\geq 0$ b. $\dfrac{x}{2x-3}<1$ c. $\dfrac{3}{x-2}\leq\dfrac{2}{x+2}$

Solution:

a. As stated above, we have more critical numbers here than in polynomial inequalities. So since the critical numbers are the values for which the numerator and denominator are zero, we can see our critical numbers are $x=-3,\,1,\,2$. Now we proceed like usual by generating a sign chart and interpreting our results. We proceed as follows



Test -4:

$$\frac{(-4+3)(-4-1)}{-4-2} = \frac{-\cdot-}{-} = \frac{+}{-} = -$$

Test 0:

$$\frac{(0+3)(0-1)}{0-2} = \frac{+\cdot-}{-} = \frac{-}{-} = +$$

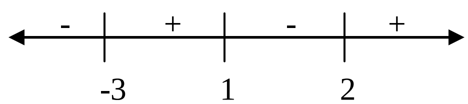
Test 1.5:

$$\frac{(1.5+3)(1.5-1)}{1.5-2} = \frac{+\cdot+}{-} = \frac{+}{-} = -$$

Test 3:

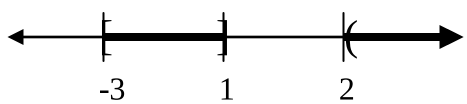
$$\frac{(3+3)(3-1)}{3-2} = \frac{+\cdot+}{+} = \frac{+}{+} = +$$

So our completed sign chart is



Since our original inequality is $\frac{(x+3)(x-1)}{x-2} \geq 0$, we want positive values and we also want the values where the rational expression is zero. Since the –3 and 1 came from the numerator, they will make the expression zero. Also, since the 2 came from the denominator and 2 is not in the domain of the expression, 2 can never be included in the solution set. Therefore our solution is

$$[-3, 1] \cup (2, \infty)$$



b. This time we must begin by getting the inequality into the proper form. That is zero on the right side and all one fraction. To do the latter we must combine the expressions over the LCD. So we proceed as follows

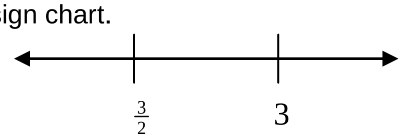
$$\frac{x}{2x-3} < 1$$

$$\frac{x}{2x-3} - 1 < 0$$

$$\frac{x}{2x-3} - \frac{2x-3}{2x-3} < 0$$

$$\frac{-x+3}{2x-3} < 0$$

Now we can see the critical numbers are $x = \frac{3}{2}, 3$. So now we can continue as before by generating our sign chart.



Test 0:

$$\frac{-0+3}{2\cdot 0-3}$$
$$=\frac{+}{-}$$
$$=-$$

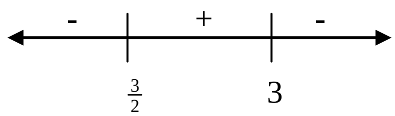
Test 2:

$$\frac{-2+3}{2\cdot 2-3}$$
$$=\frac{+}{+}$$
$$=+$$

Test 4:

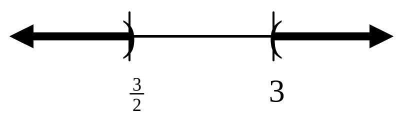
$$\frac{-4+3}{2\cdot 4-3}$$
$$=\frac{-}{+}$$
$$=-$$

So this gives us the competed sign chart of



Since our inequality is $\frac{-x+3}{2x-3}<0$ we want the negative values. Also, since the inequality symbol is a regular < sign, we don't include any of the endpoints. So our solution is

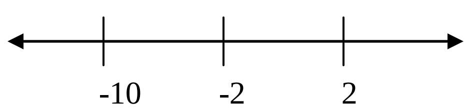
$$\left(-\infty,\ \tfrac{3}{2}\right)\cup\left(3,\ \infty\right)$$



c. Lastly, again we must put the inequality into a usable form. That is, zero on the right side one fraction. So we again combine the fractions over the LCD.

$$\frac{3}{x-2}\le\frac{2}{x+2}$$
$$\frac{3}{x-2}-\frac{2}{x+2}\le 0$$
$$\frac{3(x+2)-2(x-2)}{(x-2)(x+2)}\le 0$$
$$\frac{x+10}{(x-2)(x+2)}\le 0$$

So our critical numbers are $x=-10,\ -2,\ 2$. Now we generate the sign chart.



Test -11:

$$\frac{-11+10}{(-11-2)(-11+2)}$$
$$=\frac{-}{-\cdot -}$$
$$=\frac{-}{+}$$
$$=-$$

Test -3:

$$\frac{-3+10}{(-3-2)(-3+2)}$$
$$=\frac{+}{-\cdot -}$$
$$=\frac{+}{+}$$
$$=+$$

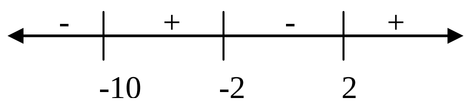
Test 0:

$$\frac{0+10}{(0-2)(0+2)}$$
$$=\frac{+}{-\cdot+}$$
$$=\frac{+}{-}$$
$$=-$$

Test 3:

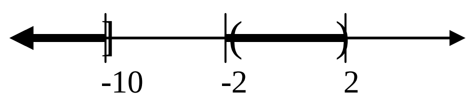
$$\frac{3+10}{(3-2)(3+2)}$$
$$=\frac{+}{+\cdot+}$$
$$=\frac{+}{+}$$
$$=+$$

So our completed sign chart is



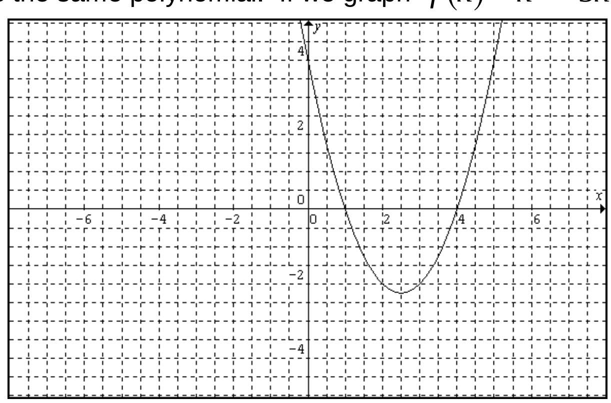
Since our original inequality is $\frac{x+10}{(x-2)(x+2)} \le 0$, we want negative values and we also want the values where the expression is zero. Since the –10 came from the numerator, it will make the expression zero. Also, since the -2 and 2 came from the denominator and they are not in the domain of the expression, they can never be included in the solution set. Therefore our solution is

$$(-\infty, -10] \cup (-2, 2)$$



Lastly we want to discuss what is happening graphically here.

Consider the inequality $x^2-5x+4<0$. As we saw in example 1, the solution to this inequality is $(1, 4)$. However, the quadratic function $f(x)=x^2-5x+4$ must be somehow related to this inequality since it contains the same polynomial. If we graph $f(x)=x^2-5x+4$ we get



So since we wanted to know when $x^2-5x+4<0$, we wanted to know when $f(x)<0$ for our *f* as graphed. To find when $f(x)<0$ means we want to know when the function

$f(x)=x^2-5x+4$ lies below the x-axis (since the points on the x-axis are the points when $f(x)=0$). From our graph we can see that $f(x)<0$ in the interval $(1,\ 4)$, just like we got when we used a sign chart.

Therefore it should be clear that the only time that a polynomial can change sign is when it crosses through the axis. That is why when constructing our sign chart we simply need to find the values at which the polynomial is zero. These values cut the graph into pieces that are entirely above or below the x-axis. Thus, the interval would be entirely positive or entirely negative.

For rational functions, the only difference is that the graph can also change from above to below the x-axis when there is an asymptote (that is, when the denominator is zero). That is why we must add the values that make both numerator and denominator zero (since when the numerator is zero the function and expression would be zero).

So, we can see an alternate method for solving a quadratic or rational function is to graph the function and then interpret the < or > symbol as the region over which the graph is below or above the x-axis. If the inequality is $f(x)<0$, then we want the x intervals over which the graph is below the x-axis, and if the inequality is $f(x)>0$, then we want the x intervals over which the graph is above the x-axis. Similarly we can find $f(x)\le 0$ and $f(x)\ge 0$ we simply need to also include the values at which the graph hits the x-axis, i.e. the values where $f(x)=0$.

Example 3:

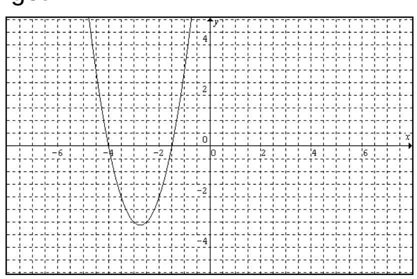
Solve by graphing the associated function.

a. $2x^2+11x+12\le 0$

b. $\dfrac{x+2}{x-3}>0$

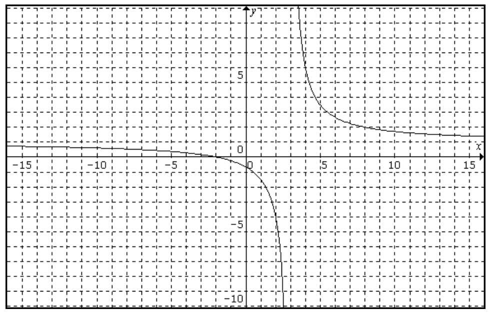
Solution:

a. We will begin by graphing the function $f(x)=2x^2+11x+12$. By using the methods learned in section 10.5 we get



Since the inequality is $2x^2+11x+12\le 0$, we are looking for where the graph is below and where it hits the x-axis. By the graph we can see that is in the interval $\left[-4,\ -1\tfrac{1}{2}\right]$.

b. Again we will begin by graphing the function $f(x)=\dfrac{x+2}{x-3}$. We can do this by plotting points or by the methods discussed in section 9.8. We get



Since this time we want to solve the inequality $\dfrac{x+2}{x-3}>0$ we want the values for which the function is above the x-axis. So we can see from the graph that would be in the interval $(3,\ \infty)$.

10.7 Exercises

Solve and graph the following.

1. $(x-2)(x+3)>0$
2. $(x-1)(x+1)<0$
3. $(x+1)(x-4)\le 0$
4. $(x+2)(x+4)\ge 0$
5. $x(x-2)\ge 0$
6. $x(x+3)\le 0$
7. $x^2(x+5)<0$
8. $x^2(x-2)>0$
9. $(x+4)^2(x-6)\ge 0$
10. $(x-6)(x+3)^2\ge 0$
11. $(x-2)(x-3)(x-4)>0$
12. $(x-1)(x+1)(x+2)\le 0$
13. $(x-1)^2(x+2)(x-2)<0$
14. $(x-4)^2(x-2)^3\ge 0$
15. $x^2-2x-3\ge 0$
16. $x^2-5x+6>0$
17. $x^2-3x\le 4$
18. $x^2-6x+9<16$
19. $2x^2+5x>12$
20. $2x^2+5x\le 3$
21. $4x^3-12x^2>0$
22. $8x^2+10x>3$
23. $4x^2+7\le 3x$
24. $x^3-4x^2-x<0$
25. $2x^3-x^4\le 0$
26. $x^4-8x<0$
27. $x^2\ge 1$
28. $x^2\le 4$
29. $x^4\ge x^2$
30. $x^3\le x^2$
31. $\dfrac{x}{x+1}>0$
32. $\dfrac{x}{x-1}<0$
33. $\dfrac{x+1}{x-3}\le 0$
34. $\dfrac{x-2}{x+3}\ge 0$
35. $\dfrac{x-2}{(x+2)(x-3)}\ge 0$
36. $\dfrac{(x+1)(x-1)}{x-4}\le 0$
37. $\dfrac{x(x-1)}{x-3}>0$
38. $\dfrac{x-7}{x(x+2)}>0$
39. $\dfrac{(x-6)(x+5)}{(x+7)(x-1)}\le 0$
40. $\dfrac{(x-3)(x-2)}{(x+3)(x+2)}\le 0$
41. $\dfrac{x^2-5x+6}{x^2-1}\ge 0$
42. $\dfrac{x^2-x-20}{x^2-3x-4}\ge 0$
43. $\dfrac{1}{x+1}<2$
44. $\dfrac{x}{x-1}>1$
45. $\dfrac{3x-5}{x-5}>4$

46. $\dfrac{2x+1}{x-1} \le 3$

47. $\dfrac{2}{x} \ge \dfrac{5}{x+6}$

48. $\dfrac{2}{x+1} \ge \dfrac{3}{x-1}$

49. $\dfrac{1}{x-3} \le \dfrac{9}{4x+3}$

50. $\dfrac{1}{x+2} \ge \dfrac{1}{x-2}$

51. $\dfrac{3}{x+2} > \dfrac{2}{x+1}$

52. $\dfrac{1}{x} < \dfrac{1}{x+3}$

53. $\dfrac{3x}{x-1} \ge \dfrac{x}{x+4}$

54. $\dfrac{2x}{x+4} \le \dfrac{5}{x}$

55. $\dfrac{x}{x-1} \le \dfrac{8}{x+2}$

56. $\dfrac{x}{x+12} < \dfrac{1}{x+5}$

57. $\dfrac{2x}{x+4} > \dfrac{3}{x-1}$

58. $\dfrac{5}{3x-8} \ge \dfrac{x}{x+2}$

59. $\dfrac{3}{x-1} - \dfrac{2}{x+1} \le 1$

60. $\dfrac{2}{x+5} - \dfrac{3}{x+3} \le \dfrac{1}{x}$

Solve by graphing the associated function.

61. $x^2 - 8x + 15 > 0$

62. $x^2 - 14x + 45 \ge 0$

63. $x^2 - 6x + 9 \le 0$

64. $x^2 - 2x - 35 < 0$

65. $2x^2 - 4x + 5 \ge 0$

66. $3x^2 \le 6x + 1$

67. $\dfrac{1}{x-1} + 1 > 0$

68. $\dfrac{1}{x-2} - 3 \le 0$

69. $\dfrac{x-1}{x+2} \le 0$

70. $\dfrac{x-2}{x+3} \ge 0$

71. $\dfrac{x+1}{x-1} \ge 0$

72. $\dfrac{2x-1}{x-4} \le 0$