10.5 Graphing Quadratic Functions

Now that we can solve quadratic equations, we want to learn how to graph the function associated with the quadratic equation. We call this the quadratic function.

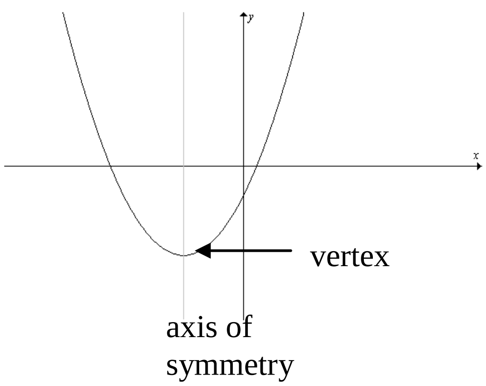
Graphs of Quadratic Functions

The graph of $f(x) = ax^2 + bx + c$ is called a parabola (U-shaped curve).

The completed square form $f(x) = a(x-h)^2 + k$ is called standard form of the function. The vertex of the parabola occurs at the point (h, k) and the vertical line $x = h$ is called the axis of symmetry.

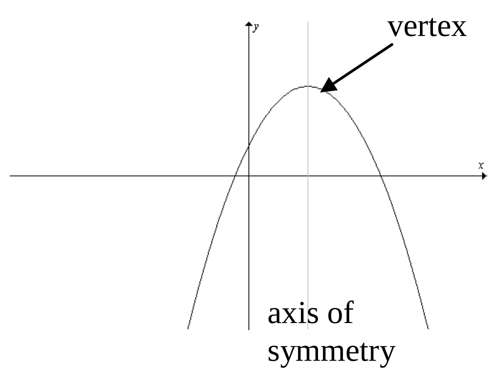
There are two possible graphs of the quadratic function.

For $a > 0$:



Opens up

For $a < 0$:



Opens down

From our basic graphs we can see the following:

1. If a is negative, $f(x) = ax^2 + bx + c$ opens down and the parabola has a maximum value of $y = k$.
2. If a is positive, $f(x) = ax^2 + bx + c$ opens up and the parabola has a minimum value of $y = k$.

All quadratic functions have a maximum or minimum and it always occurs at the vertex. For this reason, the vertex is a very important piece of information. Also notice that the value of the maximum or minimum is the y value of the vertex. The x value of the vertex just indicates the value at which that maximum or minimum occurs.

We also note that the graph is symmetry around the axis of symmetry. That is, the graph is a mirror image across the line that is the axis of symmetry.

The axis of symmetry for quadratic functions always goes through the vertex and is always the x value of the vertex.

Lastly before we begin graphing quadratic functions, we need to recall the following information.

Finding Intercepts

To find the x-intercepts, let y = 0 and solve for x.
To find the y-intercept, let x = 0 and solve for y.

Now we are ready to graph our quadratic functions.

Example 1:

Find the vertex, x- and y-intercepts, axis of symmetry, maximum or minimum values and graph the following.

a. $y = -3(x+5)^2 + 3$ b. $y = x^2 + 6x - 5$ c. $f(x) = -4x^2 - 8x + 1$

Solution:

a. First we notice that the equation is already in standard form. Therefore, using the formula above we can simply read off the vertex. Notice the signs of h and k. Standard form is set up so that we must switch the sign of the value with the x and keep the sign of the value on the outside. So for our equation $y = -3(x+5)^2 + 3$ we get the vertex is $(-5, 3)$.

Also, since the value out in front is a –3, we know that the parabola must open down. Therefore, the vertex must represent a maximum. So the maximum value is the y value of the vertex. Thus we have a maximum value of 3. Also, the axis of symmetry always goes through the x value of the vertex. Thus the axis of symmetry is $x = -5$.

So we still need the intercepts and graph of the function.

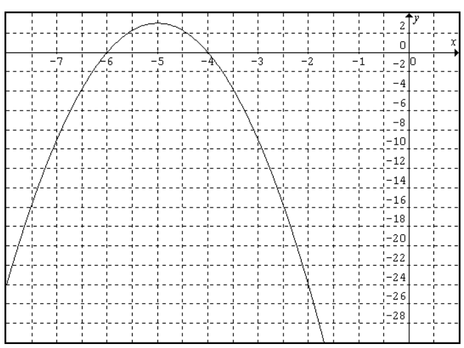
x-intercepts: (Set y=0)
Extracting roots is easiest here

$$\begin{aligned} 0 &= -3(x+5)^2 + 3 \\ -3 &= -3(x+5)^2 \\ (x+5)^2 &= 1 \\ x+5 &= \pm 1 \\ x &= -5 \pm 1 \\ x &= -6, -4 \end{aligned}$$

y-intercepts: (Set x=0)

$$\begin{aligned} y &= -3(0+5)^2 + 3 \\ &= -3(25) + 3 \\ &= -72 \end{aligned}$$

So now to graph we simply need to plot all of this given information and connect with a smooth U-shaped curve. We get



b. This time the equation is not given to us in standard form. So we must begin by putting it in standard form. To do this we need use completing the square. Recall, for completing the square, the leading coefficient had to be +1 and we add $\left(\frac{1}{2} \cdot b\right)^2$ to both sides of the equation. The only thing that is different this time is instead of adding the same value to both sides of the equation, we will add the completed square value in and subtract that same value from the same side. This way the equation will stay balanced and we will have everything on one side and y on the other side, just as standard form requires. We proceed as follows

$$\begin{aligned} y &= x^2 + 6x - 5 \\ &= \left(x^2 + 6x\right) - 5 \\ &= \left(x^2 + 6x + \left(\tfrac{1}{2}\cdot 6\right)^2\right) - 5 - \left(\tfrac{1}{2}\cdot 6\right)^2 \\ &= (x+3)^2 - 5 - 9 \\ &= (x+3)^2 - 14 \end{aligned}$$

adding $\left(\frac{1}{2}\cdot b\right)^2$

subtracting $\left(\frac{1}{2}\cdot b\right)^2$ from the same side to balance the equation

So now that the equation is in standard form we can read off the vertex, axis of symmetry and the minimum value (since the parabola is opening up because a = 1 which is positive)
Vertex: $(-3, -14)$
Axis of symmetry: $x = -3$
Minimum value is –14

Now we simply need the intercepts. We have our choice of which equation to use to find the intercepts. For the x-intercepts we will use the equation in standard form since we can easily extract the roots to solve. For the y-intercept, we will use the equation that was given to us since we can see that if we let x = 0 the equation is practically trivial.

<u>x-intercepts</u>:

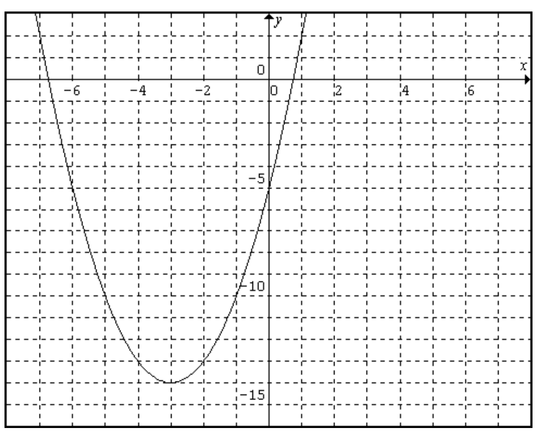
$$\begin{aligned} 0 &= (x+3)^2 - 14 \\ (x+3)^2 &= 14 \\ x+3 &= \pm\sqrt{14} \\ x &= -3 \pm \sqrt{14} \\ x &\approx -6.7,\ 0.7 \end{aligned}$$

<u>y-intercepts</u>:

$$\begin{aligned} y &= 0^2 + 6\cdot 0 - 5 \\ &= -5 \end{aligned}$$

We use decimal equivalents for the x-intercepts since we want to use them for graphing. It is clearly easier to plot 0.7 than to plot $-3+\sqrt{14}$.

Now we put it all together and graph



c. Again we must start by getting the equation to standard form. Again, remember that the leading coefficient must be +1. So we will factor the –4 out of the first two terms only (since those are the only terms we want to complete the square on) and then add our completed square piece. We proceed as follows

$$
\begin{aligned}
f(x) &= -4x^2 - 8x + 1 \\
&= -4\left(x^2 + 2x\right) + 1 \\
&= -4\left(x^2 + 2x + \left(\tfrac{1}{2}\cdot 2\right)^2\right) + 1 - \left(-4\left(\tfrac{1}{2}\cdot 2\right)^2\right) \\
&= -4(x+1)^2 + 1 + 4 \\
&= -4(x+1)^2 + 5
\end{aligned}
$$

Factor out -4 to get the leading coefficient +1

adding $\left(\frac{1}{2}\cdot b\right)^2$

subtracting $\left(\frac{1}{2}\cdot b\right)^2$

The -4 is here since what we really added in is $\left(\frac{1}{2}\cdot b\right)^2$ times the leading coefficient since it was in front of the parenthesis

So now that the equation is in standard form we can read off the vertex, axis of symmetry and the maximum value (since the parabola is opening down because a = -4)

Vertex: $(-1, 5)$

Axis of symmetry: $x = -1$

Maximum value is 5

Now we find the intercepts. Like before, for the x-intercepts we will use the equation in standard form since we can easily extract the roots and for the y-intercept, we will use the equation that was given to us since if we let x = 0 the equation very easy.

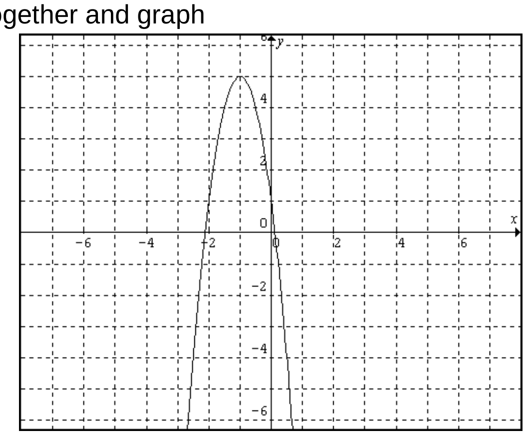
x-intercepts:

$$
\begin{aligned}
0 &= -4(x+1)^2 + 5 \\
4(x+1)^2 &= 5 \\
(x+1)^2 &= \tfrac{5}{4} \\
x + 1 &= \pm\tfrac{\sqrt{5}}{2} \\
x &= -1 \pm \tfrac{\sqrt{5}}{2} \\
x &\approx -2.1,\ 0.1
\end{aligned}
$$

y-intercepts:

$$
\begin{aligned}
y &= -4\cdot 0^2 - 8\cdot 0 + 1 \\
&= 1
\end{aligned}
$$

Now we put it all together and graph



There is another way to find the vertex which is a little easier than getting the function into standard form. Lets get the quadratic function $f(x) = ax^2 + bx + c$ into standard form. We proceed as follows

$$
\begin{aligned}
f(x) &= ax^2 + bx + c \\
&= a\left(x^2 + \tfrac{b}{a}x\right) + c \\
&= a\left(x^2 + \tfrac{b}{a}x + \left(\tfrac{1}{2}\cdot\tfrac{b}{a}\right)^2\right) + c - a\left(\tfrac{1}{2}\cdot\tfrac{b}{a}\right)^2 \\
&= a\left(x + \tfrac{b}{2a}\right)^2 + c - \tfrac{b^2}{4a}
\end{aligned}
$$

Factor out a to get the leading coefficient +1

adding $\left(\frac{1}{2}\cdot b\right)^2$

subtracting $\left(\frac{1}{2}\cdot b\right)^2$ from the same side to balance the equation while accounting for the leading coefficient

So the vertex has an x value of $x = -\frac{b}{2a}$. The y value of the vertex is really $f\left(-\frac{b}{2a}\right)$. So, we get the following

Finding the Vertex of a Quadratic Function
The vertex of $f(x) = ax^2 + bx + c$ is given by the formula $\left(-\frac{b}{2a},\ f\left(-\frac{b}{2a}\right)\right)$.

We can use either this "formula" method or the completing the square method shown above to find the vertex of a parabola.

<u>Example 2</u>:

Graph. Find the vertex, x- and y-intercepts, and any maximum or minimum values.

a. $y = -x^2 - 10x + 10$ b. $h(x) = \frac{1}{2}x^2 - x + 1$

Solution:

a. Lets use the formula $x = -\frac{b}{2a}$ to solve this problem. First, we find the vertex. So we notice that $a = -1$ and $b = -10$. Therefore the vertex is at

$$
\begin{aligned}
x &= -\tfrac{b}{2a} \\
&= -\tfrac{-10}{2(-1)} \\
&= -5
\end{aligned}
\qquad \text{and} \qquad
\begin{aligned}
f\left(-\tfrac{b}{2a}\right) &= f(-5) \\
&= -(-5)^2 - 10(-5) + 10 \\
&= -25 + 50 + 10 \\
&= 35
\end{aligned}
$$

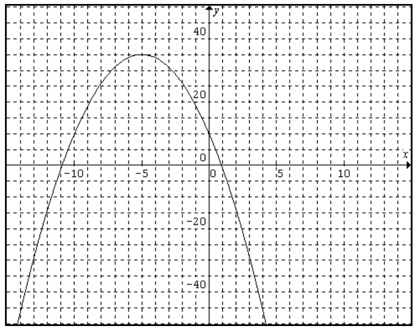
So the vertex is $(-5,\ 35)$. Since the parabola opens down, this value represents a maximum. So the maximum value is 35. Next we find the intercepts as usual. We will use the quadratic formula for the x-intercepts since in this case it is the most efficient method to use.

<u>x-intercepts</u>:

$$
\begin{aligned}
0 &= -x^2 - 10x + 10 \\
x &= \frac{-(-10) \pm \sqrt{(-10)^2 - 4(-1)(10)}}{2(-1)} \\
&= \frac{10 \pm \sqrt{140}}{-2} \\
&= \frac{10 \pm 2\sqrt{35}}{-2} \\
&= -5 \pm \sqrt{35} \\
&\approx -10.9,\ 0.9
\end{aligned}
$$

<u>y-intercept</u>:

$$
\begin{aligned}
y &= -0^2 - 10\cdot 0 + 10 \\
&= 10
\end{aligned}
$$



Now we simply plot all of our information and finish the graph.

b. Again we use the formula $x = -\frac{b}{2a}$ to solve this problem. First, we find the vertex. In this case $a = \frac{1}{2}$ and $b = -1$. Therefore the vertex is at

$$x = -\frac{b}{2a} = -\frac{-1}{2\left(\frac{1}{2}\right)} = 1$$

and

$$h\left(-\frac{b}{2a}\right) = h(1) = \frac{1}{2}(1)^2 - 1(1) + 1 = \frac{1}{2} - 1 + 1 = \frac{1}{2}$$

So the vertex is $\left(1, \frac{1}{2}\right)$. Since the parabola opens up, this value represents a minimum. So the minimum value is $\frac{1}{2}$. Next we find the intercepts as usual.

<u>x-intercepts</u>:

$$0 = \frac{1}{2}x^2 - x + 1$$

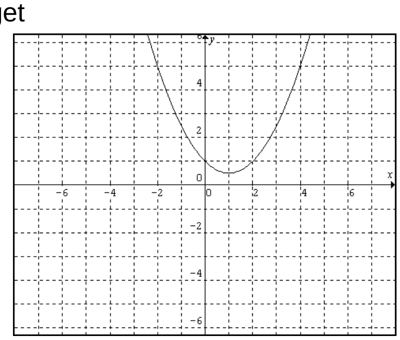
$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4\left(\frac{1}{2}\right)(1)}}{2\left(\frac{1}{2}\right)} = \frac{1 \pm \sqrt{-1}}{1} = 1 \pm i$$

<u>y-intercept</u>:

$$y = \frac{1}{2} \cdot 0^2 - 1 \cdot 0 + 1 = 1$$

Since the x-intercepts are complex numbers, the graph must have no x-intercepts. We could have also seen this by noticing that the minimum value is $\frac{1}{2}$. Since the graph can get no smaller than $\frac{1}{2}$ it cannot intercept the x-axis.

Now plot all of our information and finish the graph we can use a little symmetry to assist us in getting enough information to get a good graph. We can use the point symmetric to the y-intercept. We get



So in general, we use the following procedure for graphing a quadratic function (a parabola).

Sketching a Parabola
1. Determine the vertex by completing the square or using the formula $x = -\frac{b}{2a}$. 2. Find the x- and y-intercepts. 3. Plot the vertex and intercepts. 4. Use the fact that the parabola opens up if $a > 0$ and down if $a < 0$ and symmetry to complete the graph.

10.5 Exercises

Find the vertex, x- and y-intercepts, axis of symmetry and maximum or minimum values of the following.

1. $y = (x-1)^2 - 1$
2. $y = (x-2)^2 - 4$
3. $y = -(x+2)^2$
4. $y = 4 + 2(x-2)^2$
5. $y = 3(x-2)^2 + 1$
6. $y = -x^2 - 3$
7. $y = x^2 + 4$
8. $y = -2(x-3)^2$
9. $f(x) = (x+4)^2 + 2$
10. $f(x) = 1 - (x-1)^2$
11. $y = -(x+\frac{1}{2})^2 + 1$
12. $h(x) = (x-\frac{1}{2})^2 - 4$
13. $y = -2(x-\frac{3}{2})^2 + 18$
14. $h(x) = -(x+\frac{1}{3})^2 + 16$
15. $h(x) = 6 - \frac{1}{2}(x-3)^2$
16. $y = \frac{1}{2}(x-2)^2 - 1$
17. $y = 2(x+3)^2 - \frac{3}{2}$
18. $y = -3(x+2)^2 + \frac{2}{3}$
19. $y = -\frac{1}{3}(x-\frac{2}{3})^2 + \frac{1}{27}$
20. $y = -\frac{1}{4}(x+\frac{1}{2})^2 + \frac{3}{16}$

Graph by completing the square. Find the vertex, x- and y-intercepts, and any maximum or minimum values.

21. $h(x) = x^2 - 2x - 3$
22. $h(x) = x^2 - 4x - 5$
23. $f(x) = x^2 + 6x - 7$
24. $f(x) = x^2 + 10x + 24$
25. $g(x) = 2x^2 - 8x + 1$
26. $g(x) = 2x^2 - 12x - 5$
27. $y = -3x^2 + 12x - 2$
28. $y = -2x^2 + 4x - 1$
29. $y = 2x^2 - 8x - 7$
30. $y = 3x^2 - 6x + 9$
31. $y = x^2 - 3x$
32. $y = 2x^2 + 6x + 1$
33. $y = -2x^2 + 8x - 9$
34. $y = -3x^2 - 12x - 10$
35. $y = -x^2 + x - 1$
36. $y = -2x^2 - 4x$
37. $f(x) = -2x^2 - 3x - 4$
38. $f(x) = 3x^2 + 2x + 1$
39. $g(x) = \frac{1}{2}x^2 - \frac{1}{4}x - 2$
40. $g(x) = \frac{1}{3}x^2 + \frac{2}{3}x - 3$

Graph by using the formula $x = -\frac{b}{2a}$. Find the vertex, x- and y-intercepts, and any maximum or minimum values.

41. $h(x) = x^2 - 4x - 5$
42. $h(x) = x^2 - 2x - 3$
43. $f(x) = x^2 + 10x + 24$
44. $f(x) = x^2 + 6x - 7$
45. $g(x) = 2x^2 - 12x - 5$
46. $g(x) = 2x^2 - 8x + 1$
47. $y = -2x^2 + 4x - 1$
48. $y = -3x^2 + 12x - 2$
49. $y = 3x^2 - 6x + 9$
50. $y = 2x^2 - 8x - 7$
51. $y = 2x^2 + 6x + 1$
52. $y = x^2 - 3x$
53. $y = -3x^2 - 12x - 10$
54. $y = -2x^2 + 8x - 9$
55. $y = -2x^2 - 4x$
56. $y = -x^2 + x - 1$
57. $f(x) = 3x^2 + 2x + 1$
58. $f(x) = -2x^2 - 3x - 4$

59. $g(x)=\frac{1}{3}x^2+\frac{2}{3}x-3$

60. $g(x)=\frac{1}{2}x^2-\frac{1}{4}x-2$

Graph. Find the vertex, x- and y-intercepts, and any maximum or minimum values.

61. $y=x^2-3$

62. $y=x^2-4x+4$

63. $y=x^2-7x+4$

64. $y=x^2-5x+8$

65. $f(x)=3x^2-3x-9$

66. $g(x)=2x^2+2x+4$

67. $h(x)=-4x^2+12x+1$

68. $y=-3x^2+9x+4$

69. $y=4-8x-x^2$

70. $y=10-10x-x^2$

71. $y=-2x^2+6x$

72. $y=-x^2-4x$

73. $f(x)=\frac{1}{3}x^2-2x+1$

74. $f(x)=-\frac{1}{2}x^2-x-3$

75. $g(x)=-\frac{1}{2}x^2+x+5$

76. $g(x)=\frac{1}{3}x^2-3x+2$

77. $y=-\frac{1}{2}\left(x^2-6x+7\right)$

78. $y=-\frac{1}{4}\left(x^2+4x-1\right)$

79. $y=(2x-3)(x+1)$

80. $y=(x-3)(3x-1)$