10.4 Solving Equations in Quadratic Form, Equations Reducible to Quadratics

Now that we can solve all quadratic equations we want to solve equations that are not exactly quadratic but can either be made to look quadratic or generate quadratic equations. We start with the former.

Equations of Quadratic Form
An equation of the form $au^2 + bu + c = 0$ where u is an algebraic expression is called an equation is quadratic form.

An example of an equation in quadratic form would be $x^4 - 13x^2 + 36 = 0$. The way to visualize this is by using the properties of exponents. We could see it as

$$x^4 - 13x^2 + 36 = 0 \qquad \Rightarrow \qquad \left(x^2\right)^2 - 13\left(x^2\right) + 36 = 0$$

So the expression x^2 is the algebraic expression referred to in the above box.

Another example of an equation is $3x + 8\sqrt{x} - 3 = 0$. We can visualize it this way

$$3x + 8\sqrt{x} - 3 = 0 \qquad \Rightarrow \qquad 3\left(\sqrt{x}\right)^2 + 8\sqrt{x} - 3 = 0$$

This time the expression is $\sqrt{x}$.

We generally find the expression involved by looking for something that shows up to the first power and then also to the second.

We solve these equations by using a simple substitution procedure. We will explain this by way of example.

Example 1:

Find all solutions to the following equations.

a. $x^4 - 13x^2 + 36 = 0$ b. $3x + 8\sqrt{x} - 3 = 0$

Solution:

a. By looking at the work we did above we know

$$x^4 - 13x^2 + 36 = 0 \qquad \Rightarrow \qquad \left(x^2\right)^2 - 13\left(x^2\right) + 36 = 0$$

So we make a substitution as follows: Let $u = x^2$. If we do this, then by substituting this into the equation on the right we get

$$u^2 - 13u + 36 = 0$$

This is clearly an equation that we can solve. It's just quadratic. Therefore we will solve. Lets do so by factoring. We get

$$u^2 - 13u + 36 = 0$$
$$(u-9)(u-4) = 0$$
$$u = 9 \quad \& \quad u = 4$$

However, the original equation was in terms of x. Therefore we must solve for x. So we recall our substitution $u = x^2$. Re-substituting we have

$$x^2 = 9 \quad \& \quad x^2 = 4$$

Now we simply solve by extracting roots. So our answer is

$$x = \pm 3 \quad \& \quad x = \pm 2$$

So our solution set is $\{-3, -2, 2, 3\}$.

b. Again by recalling the work done above we have

$$3x + 8\sqrt{x} - 3 = 0 \qquad \Rightarrow \qquad 3\left(\sqrt{x}\right)^2 + 8\sqrt{x} - 3 = 0$$

So again, our substitution is $u = \sqrt{x}$. Subbing this into the equation on the right we have

$$3u^2 + 8u - 3 = 0$$

We will solve by factoring

$$3u^2 + 8u - 3 = 0$$
$$(3u - 1)(u + 3) = 0$$
$$u = \tfrac{1}{3} \quad \& \quad u = -3$$

Re subbing $u = \sqrt{x}$ and solving like before we get

$$\sqrt{x} = \tfrac{1}{3} \quad \& \quad \sqrt{x} = -3$$
$$x = \tfrac{1}{9} \quad \& \quad x = 9$$

However, recall that when squaring both sides of an equation, we must check our answer for extraneous roots. So we go to the original equation and check them there.

Check $x = \frac{1}{9}$:

$$3\left(\tfrac{1}{9}\right) + 8\sqrt{\tfrac{1}{9}} - 3 = 0$$
$$\tfrac{1}{3} + \tfrac{8}{3} - 3 = 0$$
$$\tfrac{9}{3} - 3 = 0$$

$x = \frac{1}{9}$ checks

Check $x = 9$:

$$3(9) + 8\sqrt{9} - 3 = 0$$
$$27 + 24 - 3 = 0$$
$$48 = 0$$

$x = 9$ does not check

Since the $x = 9$ does not check it is extraneous and thus thrown out.
So the solution set is $\left\{\frac{1}{9}\right\}$.

So by the example we get the following method

Solving an Equation in Quadratic Form
1. Determine the substitution to be used 2. Make the substitution and solve the resulting equation. 3. Re-substitute the expression from step 1 and solve for the original variable. 4. Check your answer if necessary.

Lets see some more examples.

Example 2:

Find all solutions to the following equations.

a. $z^{2/3} - z^{1/3} - 6 = 0$ b. $2x^{-2} = x^{-1} - 1$ c. $\left(x^2 - 6x\right)^2 - 2\left(x^2 - 6x\right) - 35 = 0$

Solution:

a. We need to first determine the substitution. We are looking for an expression in the equation that is both squared and to the first power. Notice that $\left(z^{1/3}\right)^2 = z^{2/3}$ by properties of exponents. So therefore we use the substitution $u = z^{1/3}$. So we have

$$z^{2/3} - z^{1/3} - 6 = 0$$
$$\left(z^{1/3}\right)^2 - z^{1/3} - 6 = 0$$
$$u^2 - u - 6 = 0$$

So we now solve the quadratic, re-sub and solve for z as follows

$$\begin{aligned} u^2 - u - 6 &= 0 \\ (u-3)(u+2) &= 0 \\ u = 3 \quad &\& \quad u = -2 \\ z^{1/3} = 3 \quad &\& \quad z^{1/3} = -2 \qquad \text{Re}-sub\ u = z^{1/3} \\ z = 27 \quad &\& \quad z = -8 \qquad Cubing\ both\ sides \end{aligned}$$

Checking our answers reveals that they are both good answers.
So our solution set is $\{-8, 27\}$.

b. Again, we start by determining the substitution we should use. Notice $\left(x^{-1}\right)^2 = x^{-2}$.
So let $u = x^{-1}$ and solve as above. We get

$$\begin{aligned} 2x^{-2} &= x^{-1} - 1 \\ 2u^2 &= u - 1 \\ 2u^2 - u + 1 &= 0 \\ (2u+1)(u-1) &= 0 \\ u = -\tfrac{1}{2} \quad &\& \quad u = 1 \\ x^{-1} = -\tfrac{1}{2} \quad &\& \ x^{-1} = 1 \end{aligned}$$

So we need to solve the bottom equations. To do this we will get rid of the negative exponents and then solve by clearing fractions.

$$\begin{aligned} x^{-1} = -\tfrac{1}{2} \quad &\& \quad x^{-1} = 1 \\ \tfrac{1}{x} = -\tfrac{1}{2} \quad &\& \quad \tfrac{1}{x} = 1 \\ x = -2 \quad &\& \quad x = 1 \end{aligned}$$

So our solution set is $\{-2, 1\}$.

c. This time it is very clear what our substitution should be. Clearly let $u = x^2 - 6x$. Subbing and solving like before and solving we have

$$
\begin{aligned}
\left(x^2-6x\right)^2-2\left(x^2-6x\right)-35&=0\\
u^2-2u-35&=0\\
(u-7)(u+5)&=0
\end{aligned}
$$

$$
\begin{array}{ccc}
u=7 & & u=-5\\
x^2-6x=7 & & x^2-6x=-5\\
x^2-6x-7=0 & \& & x^2-6x+5=0\\
(x-7)(x+1)=0 & & (x-5)(x-1)=0\\
x=7 \quad \& \quad x=-1 & & x=5 \quad \& \quad x=1
\end{array}
$$

So our solution set is $\{-1, 1, 5, 7\}$.

The other type of equation we wanted to solve was equations that generate quadratic equations. This usually happens on radical or rational equations. Since we have discussed solving these types previously, we will merely refresh our memories on the techniques used.

<u>Example 3</u>:

Find all solutions to the following equations.

a. $\sqrt{5-2x}-\sqrt{2-x}=1$ b. $\dfrac{2}{y+1}+\dfrac{1}{y-1}=1$

Solution:

a. To solve this equation we recall that to solve a radical equation we must isolate a radical and square both sides as many times as needed until all the radicals are removed (c.f. Section 8.7). So we do that process here.

$$
\begin{aligned}
\sqrt{5-2x}-\sqrt{2-x}&=1\\
\sqrt{5-2x}&=\sqrt{2-x}+1\\
\left(\sqrt{5-2x}\right)^2&=\left(\sqrt{2-x}+1\right)^2\\
5-2x&=2-x+2\sqrt{2-x}+1\\
2\sqrt{2-x}&=2-x\\
\left(2\sqrt{2-x}\right)^2&=(2-x)^2\\
4(2-x)&=4-4x+x^2\\
8-4x&=4-4x+x^2\\
x^2-4&=0\\
x&=\pm 2
\end{aligned}
$$

Again, since we raised both sides to an even power, we must check our answers.

Check $x = 2$:

$$\sqrt{5-2(2)} - \sqrt{2-(2)} = 1$$
$$\sqrt{5-4} - \sqrt{2-2} = 1$$
$$\sqrt{1} - \sqrt{0} = 1$$
$$1 = 1$$

Check $x = 2$:

$$\sqrt{5-2(-2)} - \sqrt{2-(-2)} = 1$$
$$\sqrt{5+4} - \sqrt{2+2} = 1$$
$$\sqrt{9} - \sqrt{4} = 1$$
$$3 - 2 = 1$$

Both solutions check. So our solution set is $\{-2,\ 2\}$.

b. To solve this equation we recall that to solve a rational equation we must multiply both sides by the least common denominator then solve the remaining equation (c.f. Section 5.5). In this case the LCD is $(y+1)(y-1)$. So we solve as follows

$$\frac{2}{y+1} + \frac{1}{y-1} = 1$$
$$(y+1)(y-1)\left(\frac{2}{y+1} + \frac{1}{y-1}\right) = (1)(y+1)(y-1)$$
$$2(y-1) + (y+1) = (y+1)(y-1)$$
$$2y - 2 + y + 1 = y^2 - 1$$
$$y^2 - 3y = 0$$
$$y(y-3) = 0$$
$$y = 0,\ 3$$

Since we started with a rational equation (which has a limited domain) we must make sure that we have no extraneous solutions. To do this we simply need to check that neither solution makes the origonal equation undefined. That is, does either solution give the origonal equation a zero in the denominator.

Since it is clear that the only values that make the denominator zero are –1 and +1, our solutions are not extraneous.

Thus our solution set is $\{0,\ 3\}$.

Notice that the quadratic equations in our examples all factored nicely. If the equation does not factor simply use the quadratic formula to solve. If the equation required a substitution to solve, be sure to re-substitute and finish solving after using the quadratic formula. To check the answers simply use a decimal approximation.

10.4 Exercises

Find all solutions to the following equations.

1. $x^4 - 25x^2 + 144 = 0$
2. $x^4 - 5x^2 + 4 = 0$
3. $x + 7\sqrt{x} + 10 = 0$
4. $x + 7\sqrt{x} + 6 = 0$
5. $z^{2/3} - 8z^{1/3} + 7 = 0$
6. $a^{2/3} - 4a^{1/3} - 12 = 0$
7. $y^{-2} - 6y^{-1} + 9 = 0$
8. $d^{-2} - 7d^{-1} + 10 = 0$
9. $2x + 7x^{1/2} - 4 = 0$
10. $3x + x^{1/2} - 4 = 0$

11. $20x^4 - 25x^2 + 5 = 0$

12. $4x^4 - x^2 - 5 = 0$

13. $6x^{1/2} - 23x^{1/4} + 7 = 0$

14. $3x^{1/2} - 4x^{1/4} + 1 = 0$

15. $x^{2/5} - 2x^{1/5} + 1 = 0$

16. $x^{2/5} + 2x^{1/5} + 1 = 0$

17. $x^3 = x^{3/2} + 6$

18. $2x^{2/3} = 15 + x^{1/3}$

19. $x^{1/3} + x^{1/6} = 2$

20. $5x^{1/3} = 4x^{1/6} + 9$

21. $4y^{-2} + 1 + 3y^{-4} = 0$

22. $21t^{-2} + 18t^{-4} = 9$

23. $x - 2\sqrt{x} - 6 = 0$

24. $x - 4\sqrt{x} - 1 = 0$

25. $t^{-2} + 4 = 6t^{-1}$

26. $y^{-2} = 3 - y^{-1}$

27. $5x^4 - 7x^2 + 1 = 0$

28. $3x^4 + 5x^2 = 1$

29. $2x^{1/3} + 7 = 3x^{2/3}$

30. $x^{1/3} - x^{2/3} = 5$

31. $(x^2 - 1)^2 + (x^2 - 1) - 6 = 0$

32. $\frac{1}{x^2} - \frac{3}{x} + 2 = 0$

33. $3\left(\frac{x}{x+1}\right)^2 + 7\left(\frac{x}{x+1}\right) - 6 = 0$

34. $4\left(\frac{x+1}{x-1}\right)^2 + 19\left(\frac{x+1}{x-1}\right) - 5 = 0$

35. $(\sqrt{x} - 1)^2 + 3(\sqrt{x} - 1) - 4 = 0$

36. $(x^2 - 3x)^2 - 10(x^2 - 3x) + 24 = 0$

37. $9\left(\frac{x+2}{x+1}\right)^2 - 6\left(\frac{x+2}{x+1}\right) + 1 = 0$

38. $(3 - \sqrt{x})^2 - 10(3 - \sqrt{x}) + 23 = 0$

39. $x^6 + 7x^3 - 8 = 0$

40. $x^6 - 7x^3 - 8 = 0$

41. $\sqrt{x} + 2 = x$

42. $\sqrt{8x+1} = x + 2$

43. $\sqrt{2(x+5)} = x + 5$

44. $\sqrt{1+6x} = 2 - \sqrt{6x}$

45. $x + 3\sqrt{x-2} = 12$

46. $4\sqrt{x+1} - x = 1$

47. $3 + \sqrt{5-x} = x$

48. $x = \sqrt{x-1} + 3$

49. $\sqrt{x+2} + \sqrt{3x+4} = 2$

50. $\sqrt{6x+7} - \sqrt{3x+3} = 1$

51. $\sqrt{x} + \sqrt{x-9} = 1$

52. $\sqrt{x} + \sqrt{x-5} = 5$

53. $\sqrt{x-2} - \sqrt{4x+1} + 3 = 0$

54. $\sqrt{2x+7} - \sqrt{3x-1} = 2$

55. $\sqrt{3x+1} + x = 4$

56. $x - \sqrt{7x+1} = 6$

57. $\sqrt{2x+1} + \sqrt{3x-1} = 2$

58. $1 - \sqrt{1-x} + \sqrt{1-2x} = 0$

59. $\sqrt{x} + \sqrt{2x} + \sqrt{3x} = 0$

60. $\sqrt{4x} + \sqrt{2x} - \sqrt{x} = 0$

61. $x + \frac{6}{x} = -5$

62. $x + \frac{5}{x} = -6$

63. $\frac{2x}{x+4} = \frac{3}{x-1}$

64. $\frac{5}{3n-8} = \frac{n}{n+2}$

65. $x - \frac{6}{x-3} = \frac{2x}{x-3}$

66. $\frac{3x}{x+1} = \frac{12}{x^2-1} + 2$

67. $\frac{3x}{x-4}=\frac{2x}{x-3}+\frac{6}{x^2-7x+12}$

68. $\frac{x}{x-2}+\frac{1}{x-4}=\frac{2}{x^2-6x+8}$

69. $\frac{8}{y}=\frac{2}{y-2}+1$

70. $\frac{5}{y-3}-\frac{30}{y^2-9}=1$

71. $1+\frac{x-1}{x-3}=\frac{2}{x-3}-x$

72. $7-\frac{x-2}{x+3}=\frac{x^2-4}{x+3}+5$

73. $\frac{x}{x-1}+\frac{x}{x+1}=1$

74. $\frac{2}{x+2}+\frac{3}{x-3}=2$

75. $\frac{1}{x+2}-\frac{1}{x-2}=1$

76. $\frac{x}{2x+1}-\frac{2x}{x-3}=3$

77. $\frac{x-1}{x+1}+\frac{x+1}{x-1}=1$

78. $\frac{x+3}{x-1}+\frac{x}{2x+3}=7$

79. $\frac{2x+1}{3x}-\frac{x+1}{x^2}=4$

80. $\frac{x-7}{2x^2}-7=\frac{1}{x}$