10.3 The Quadratic Formula

We mentioned in the last section that completing the square can be used to solve any quadratic equation. So we can use it to solve $ax^2+bx+c=0$. We proceed as follows

$$ax^2+bx+c=0$$

$$\frac{ax^2}{a}+\frac{bx}{a}+\frac{c}{a}=\frac{0}{a}$$

$$x^2+\frac{b}{a}x=-\frac{c}{a}$$

$$x^2+\frac{b}{a}x+\left(\frac{1}{2}\cdot\frac{b}{a}\right)^2=\left(\frac{1}{2}\cdot\frac{b}{a}\right)^2-\frac{c}{a}$$

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a}$$

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$

$$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

$$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

The last line of this we call the quadratic formula.

The Quadratic Formula
The solutions to the quadratic equation $ax^2+bx+c=0$ are given by $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$.

Since we got this formula by completing the square, and completing the square always works, the quadratic formula will always work. Also, since it is much easier to use than completing the square (as we will see later), solving quadratic formulas with the quadratic formula is the preferred method.

There is one special part of the quadratic formula that can help us determine the type of solutions we have. It's the expression b^2-4ac. We call it the discriminant.

The Discriminant b^2-4ac
1. If $b^2-4ac > 0$, the equation has two distinct real number solutions. 2. If $b^2-4ac < 0$, the equation has two distinct complex number solutions. 3. If $b^2-4ac = 0$, the equation has one real solution given by $x=-\frac{b}{2a}$.

This seems fairly clear if we just notice that the discriminant is the expression under the radical. When the value under a radical is positive we get real numbers and when its negative we get complex numbers. Thus that characterizes the solutions. For the last case, if the value under the radical is zero, since $\sqrt{0}=0$ and $\pm 0=0$ we lose the whole back part of the quadratic formula and just get left with the value $x=-\frac{b}{2a}$ which is one real number value.

<u>Example 1</u>:

Determine the type of solutions the equations have

a. $x^2-5x-2=0$ b. $2x^2-x+2=0$ c. $4x^2+4x+1=0$

Solution:

a. In order to determine the type of solutions to the equation we must evaluate the discriminant. So to do that we must first determine the values of *a*, *b* and *c*. Since our equation is in standard form (everything on one side, zero on the other side) we can simply read these values off they are always in order when the equation is in standard form.
Since our equation is $x^2-5x-2=0$, we can see $a=1$, $b=-5$ and $c=-2$. So we evaluate the discriminant as follows

$$\begin{aligned} b^2-4ac &= (-5)^2-4(1)(-2) \\ &= 25+8 \\ &= 33 \end{aligned}$$

Since the value is positive, we know that the equation must have two real number solutions.

b. Again, the equation is already in standard form. So $a=2$, $b=-1$ and $c=2$.
So evaluating the discriminant we have

$$\begin{aligned} b^2-4ac &= (-1)^2-4(2)(2) \\ &= 1-16 \\ &= -15 \end{aligned}$$

Since the value is negative, we know that the equation must have two complex number solutions.

c. Finally, we see $a=4$, $b=4$ and $c=1$. So

$$\begin{aligned} b^2-4ac &= (4)^2-4(4)(1) \\ &= 16-16 \\ &= 0 \end{aligned}$$

Since the value is zero, we know that the equation must have one real number solution.
Furthermore, we know that solution is $x=-\frac{b}{2a}=-\frac{4}{2(4)}=-\frac{1}{2}$.

Lets do some solving with the quadratic formula.

<u>Example 2</u>:

Solve the following using the quadratic formula.

a. $2x^2-x+1=0$ b. $-5u^2-15u=-10$ c. $y(1-y)=1-6y^2$

Solution:

To solve using the quadratic formula we first need the equation in standard form. Then, similar to the discriminant examples, we can simply read the *a*, *b* and *c* values off of the equation since the come in order.

a. Notice that this equation is already in standard form. So since our equation is $2x^2 - x + 1 = 0$ we can see clearly that $a = 2$, $b = -1$ and $c = 1$. We now simply need to insert the values into the quadratic formula and simplify the expression as much as possible. We get

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(1)}}{2(2)} \\ &= \frac{1 \pm \sqrt{1-8}}{4} \\ &= \frac{1 \pm \sqrt{-7}}{4} \\ &= \frac{1 \pm i\sqrt{7}}{4} \\ &= \frac{1}{4} \pm i\frac{\sqrt{7}}{4} \end{aligned}$$

So our solution set is $\left\{ \frac{1}{4} - i\frac{\sqrt{7}}{4}, \frac{1}{4} + i\frac{\sqrt{7}}{4} \right\}$.

b. First we will put our equation into standard form by moving the 10 over. We get

$$-5u^2 - 15u + 10 = 0$$

Now we can see $a = -5$, $b = -15$ and $c = 10$. Plugging these values into the quadratic formula gives us

$$\begin{aligned} u &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-15) \pm \sqrt{(-15)^2 - 4(-5)(10)}}{2(-5)} \\ &= \frac{15 \pm \sqrt{225 + 200}}{-10} \\ &= \frac{15 \pm \sqrt{425}}{-10} \\ &= \frac{15 \pm 5\sqrt{17}}{-10} \\ &= \frac{15}{-10} \pm \frac{5\sqrt{17}}{-10} \end{aligned}$$

$$= -\frac{3}{2} \pm \frac{\sqrt{17}}{2}$$

So our solution set is $\left\{ -\frac{3}{2} - \frac{\sqrt{17}}{2}, -\frac{3}{2} + \frac{\sqrt{17}}{2} \right\}$.

Notice on this problem that the variable for which we were solving was *u*, therefore, when using the quadratic formula we had to use

$u = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ instead of $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. It's a subtle difference but an important one we will see in the example 3.

c. Finally, we start by getting the equation into standard form as follows

$$y(1-y) = 1 - 6y^2$$
$$y - y^2 = 1 - 6y^2$$
$$5y^2 + y - 1 = 0$$

So we have $a = 5$, $b = 1$ and $c = -1$. Plugging into the quadratic formula gives

$$\begin{aligned} y &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(1) \pm \sqrt{(1)^2 - 4(5)(-1)}}{2(5)} \\ &= \frac{-1 \pm \sqrt{1+20}}{10} \\ &= \frac{-1 \pm \sqrt{21}}{10} \\ &= -\frac{1}{10} \pm \frac{\sqrt{21}}{10} \end{aligned}$$

So our solution set is $\left\{ -\frac{1}{10} - \frac{\sqrt{21}}{10}, -\frac{1}{10} + \frac{\sqrt{21}}{10} \right\}$.

Finally, we can also use the quadratic formula to solve quadratic equations that have more than one variable involved (called Literal Equations).

To do this we just treat all other variables as if they we just numbers and solve using whichever technique seems fitting. If we decide on using the quadratic formula we must be very careful about how we choose the values of *a*, *b* and *c*.

<u>Example 3</u>:

Solve the equations for the specified variable.

a. $A = P(1+r)^2$ for *r*

b. $A = 2\pi rh + 2\pi r^2$ for *r*

Solution:

a. For this equation, since we are solving for r it seems that extracting roots would be the most efficient method. So we will proceed as follows

$$A = P(1+r)^2$$
$$(1+r)^2 = \frac{A}{P}$$
$$1+r = \pm\sqrt{\frac{A}{P}}$$
$$r = -1 \pm \sqrt{\frac{A}{P}}$$

So $r = -1 \pm \sqrt{\frac{A}{P}}$.

b. This time, the most direct method is the quadratic formula. So we start by getting the equation into standard form. That is

$$A = 2\pi rh + 2\pi r^2$$
$$2\pi r^2 + 2\pi rh - A = 0$$

The variable we are trying to solve for is r. So we need to treat everything else like a constant. So it might be useful for us to visualize the equation as

$$(2\pi)r^2 + (2\pi h)r - A = 0$$

This way we can see that $a = 2\pi$, $b = 2\pi h$ and $c = -A$. So into the quadratic formula we get

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(2\pi h) \pm \sqrt{(2\pi h)^2 - 4(2\pi)(-A)}}{2(2\pi)}$$
$$= \frac{-2\pi h \pm \sqrt{4\pi^2 h^2 + 8\pi A}}{4\pi}$$
$$= \frac{-2\pi h \pm \sqrt{4(\pi^2 h^2 + 2\pi A)}}{10}$$
$$= \frac{-2\pi h \pm 2\sqrt{\pi^2 h^2 + 2\pi A}}{10}$$
$$= \frac{-\pi h \pm \sqrt{\pi^2 h^2 + 2\pi A}}{5}$$

So, $r = \frac{-\pi h \pm \sqrt{\pi^2 h^2 + 2\pi A}}{5}$.

In this example we can see the importance of correctly labeling the variable for the quadratic formula since there are a number of variables involved in the problem.

10.3 Exercises

Determine the type of solutions the each of the following equations has.

1. $x^2+7x-2=0$
2. $x^2-7x+4=0$
3. $x^2+x+2=0$
4. $x^2+x+1=0$
5. $3x^2-18x+5=0$
6. $3x^2-8x+2=0$
7. $36x^2+84x+49=0$
8. $4x^2-12x+9=0$
9. $x^2+13=4x$
10. $x^2+3x=8$

Solve the following using the quadratic formula.

11. $x^2-6x-2=0$
12. $x^2+5x-3=0$
13. $a^2+4a-8=0$
14. $x^2-3x+4=0$
15. $2x^2-5x-2=0$
16. $3u^2-u+6=0$
17. $4x^2+10x-4=0$
18. $5x^2+33x-14=0$
19. $6y^2+6=13y$
20. $2x^2+7x=1$
21. $5b^2=3b-2$
22. $3x^2=-2x-1$
23. $3x^2=6x+2$
24. $2x^2=3-x$
25. $x^2+13=2x$
26. $(v-2)(2v-3)=5v+1$
27. $(x+2)(x-3)=x-2$
28. $(t-4)(t-2)=3t-12$
29. $(p-1)^2=2p^2-3p$
30. $(y+2)^2=y-2$
31. $(x-1)^2=(2x-5)^2$
32. $(x+2)^2=(3x-2)^2$
33. $(3x-1)^2-(5x-3)^2=0$
34. $(2x+1)^2-(3x-2)^2=0$
35. $x^2-0.3x+0.2=0$
36. $0.2x^2+0.4x+0.03=0$
37. $0.21x^2-x+0.22=0$
38. $\frac{1}{3}x^2+\frac{1}{4}x-2=0$
39. $\frac{2}{3}x^2-\frac{1}{2}x-1=0$
40. $\frac{4}{5}x^2-\frac{1}{3}x+\frac{1}{2}=0$

Solve by any appropriate method.

41. $x^2-18x=9$
42. $x^2-36x=0$
43. $0.3y^2-2y+0.1=0$
44. $1.35+0.24u+0.01u^2=0$
45. $28=x+2x^2$
46. $2z^2-4z+7=9$
47. $(x+3)(2x+1)=-3$
48. $(x-2)(x+9)=10$
49. $(v-3)^2=250$
50. $(x-5)^2+4=0$
51. $4(y+1)^2-27=0$
52. $25(x-3)^2-36x=0$
53. $2y(y-18)+3(y-18)=0$
54. $x(x+5)-10(x+5)=0$

Solve for the indicated variable.

55. $A=6s^2$ for s
56. $a^2+b^2=c^2$ for b
57. $a^2+b^2+c^2=d^2$ for c
58. $s=v_0t+\dfrac{gt^2}{2}$ for t
59. $N=\dfrac{k^2-3k}{2}$ for k
60. $A=\pi r^2+\pi rs$ for r
61. $N=\frac{1}{2}(n^2-n)$ for n
62. $A=A_0(1-r)^2$ for r
63. $2A+T=3T^2$ for T
64. $V=\frac{1}{3}\pi(R^2-r^2)$ for r
65. $(x-h)^2+(y-k)^2=r^2$ for x
66. $\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=1$ for y
67. $y=a(x-h)^2+k$ for x
68. $d^2=(x_2-x_1)^2+(y_2-y_1)^2$ for x_2