

10.2 Solving Quadratic Equations by Completing the Square

Consider the equation

$$(x-3)^2 = 5$$

We can see clearly that the solutions are

$$x = 3 \pm \sqrt{5}$$

However,

What if the equation was given to us in standard form, that is

$$x^2 - 6x + 4 = 0$$

How would we go about solving the equation? What we want to do is change the equation from standard form to the extracting roots form. That way we can easily solve. To do this we use something called completing the square.

Completing the Square

To complete the square for the expression $x^2 + bx$ we add $(\frac{1}{2} \cdot b)^2$. We get

$$x^2 + bx + (\frac{1}{2} \cdot b)^2 = (x + \frac{1}{2} \cdot b)^2$$

Notice that when we add this completing the square piece, we get a perfect square trinomial. This will always be the case, therefore, we will always be able to factor the expression once we have added in the expression $(\frac{1}{2} \cdot b)^2$.

Example 1:

Complete the square on $y^2 + 2y$. Then factor accordingly.

Solution:

We simply add on the $(\frac{1}{2} \cdot b)^2$ piece. In this case $b = 2$. So we get

$$y^2 + 2y + (\frac{1}{2} \cdot 2)^2 = y^2 + 2y + 1$$

Factoring we get

$$y^2 + 2y + 1 = (y + 1)^2$$

Notice that when we complete the square, the resulting trinomial will always factor the same way. That is, ("variable" +/- "inside () of completed square piece")². We know if its + or - by taking the same sign as the original b value. So example 1 we can visualize like the following

$$\begin{array}{c}
 \text{variable} \\
 \downarrow \quad \downarrow \\
 y^2 + 2y + (\frac{1}{2} \cdot 2)^2 = (y + 1)^2 \\
 \uparrow \quad \uparrow \quad \uparrow \\
 \text{sign} \quad \text{inside the parenthesis}
 \end{array}$$

Again, when we complete the square it will always factor like this. So we merely need to remember this pattern for more complicated problems.

So we really want to use completing the square to solve quadratic equations. To do that we first will notice a few things. To complete the square, the leading coefficient must always be +1 as it is above (if its not we can easily make it +1) and completing the square will work for any quadratic equation. This means that we can always use completing the square as a technique for solving.

Solving $ax^2 + bx + c = 0$ by Completing the Square

1. Isolate the variable terms and make the leading coefficient +1 by dividing each term by a .
2. To preserve the equality, add the expression $(\frac{1}{2} \cdot b)^2$ to both sides of the equation.
3. Factor the variable side of the equation. It will always factor as a perfect square.
4. Solve by extracting roots.

Example 2:

Solve the equations by completing the square.

a. $x^2 - 6x + 7 = 0$

b. $x^2 + \frac{4}{5}x - 1 = 0$

c. $3x^2 - 24x - 5 = 0$

Solution:

- a. To solve by completing the square we follow the steps above. We get

$$\begin{array}{ll}
 x^2 - 6x + 7 = 0 & \text{Isolate the variables} \\
 x^2 - 6x = -7 & \\
 x^2 - 6x + (\frac{1}{2} \cdot 6)^2 = -7 + (\frac{1}{2} \cdot 6)^2 & \text{Add } (\frac{1}{2} \cdot b)^2 \text{ to both sides} \\
 (x - 3)^2 = -7 + 9 & \text{Factor as shown above} \\
 (x - 3)^2 = 2 & \text{Finish by extracting roots} \\
 x - 3 = \pm\sqrt{2} & \\
 x = 3 \pm \sqrt{2} &
 \end{array}$$

So the solution set is $\{3 - \sqrt{2}, 3 + \sqrt{2}\}$.

- b. Again, we proceed as follows

$$\begin{array}{l}
 x^2 + \frac{4}{5}x - 1 = 0 \\
 x^2 + \frac{4}{5}x = 1 \\
 x^2 + \frac{4}{5}x + (\frac{1}{2} \cdot \frac{4}{5})^2 = 1 + (\frac{1}{2} \cdot \frac{4}{5})^2 \\
 (x + \frac{2}{5})^2 = 1 + \frac{4}{25} \\
 (x + \frac{2}{5})^2 = \frac{29}{25} \\
 x + \frac{2}{5} = \pm\sqrt{\frac{29}{25}} \\
 x = -\frac{2}{5} \pm \frac{\sqrt{29}}{5}
 \end{array}$$

So the solution set is $\{-\frac{2}{5} - \frac{\sqrt{29}}{5}, -\frac{2}{5} + \frac{\sqrt{29}}{5}\}$.

Note: The previous question shows us the advantage to using the formula

("variable" +/- "inside () of completed square piece")² to factor the trinomial portion. It simplifies the factoring which would have been much more difficult in this case. So we can instead just remember the pattern and use it to factor all completing the square problems no matter how difficult or trivial.

- c. Finally, we will again complete the square to solve. However, this time we need to make sure the leading coefficient is +1. So, we will start by dividing each term by 3 and then proceed as usual.

$$\begin{aligned}
 3x^2 - 24x - 5 &= 0 \\
 \frac{3x^2}{3} - \frac{24x}{3} - \frac{5}{3} &= \frac{0}{3} \\
 x^2 - 8x - \frac{5}{3} &= 0 \\
 x^2 - 8x &= \frac{5}{3} \\
 x^2 - 8x + \left(\frac{1}{2} \cdot 8\right)^2 &= \frac{5}{3} + \left(\frac{1}{2} \cdot 8\right)^2 \\
 (x - 4)^2 &= \frac{5}{3} + 16 \\
 (x - 4)^2 &= \frac{53}{3} \\
 x - 4 &= \pm \sqrt{\frac{53}{3}} \\
 x &= 4 \pm \frac{\sqrt{159}}{3}
 \end{aligned}$$

In the last step, the radical was simplified by rationalizing the denominator.

So our solution set is $\left\{4 - \frac{\sqrt{159}}{3}, 4 + \frac{\sqrt{159}}{3}\right\}$.

Now that we have the basic idea of completing the square, let's see some examples which are a little more challenging.

Example 3:

Solve the equations by completing the square.

a. $1 - x - x^2 = 0$

b. $8u - 5 = (u - 4)(u - 2)$

Solution:

- a. The first thing we must do is write the equation in standard form. So we move the terms around to get

$$-x^2 - x + 1 = 0$$

So in this equation it's really easy to just start completing the square without remembering that we need the leading coefficient to be +1. It is currently -1. So we can fix that by multiplying the entire equation by -1. We then proceed as normal.

$$-x^2 - x + 1 = 0$$

$$x^2 + x - 1 = 0$$

$$x^2 + x = 1$$

$$x^2 + x + \left(\frac{1}{2} \cdot 1\right)^2 = 1 + \left(\frac{1}{2} \cdot 1\right)^2$$

$$\left(x + \frac{1}{2}\right)^2 = 1 + \frac{1}{4}$$

$$\left(x + \frac{1}{2}\right)^2 = \frac{5}{4}$$

$$x + \frac{1}{2} = \pm\sqrt{\frac{5}{4}}$$

$$x = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}$$

So our solution set is $\left\{-\frac{1}{2} - \frac{\sqrt{5}}{2}, -\frac{1}{2} + \frac{\sqrt{5}}{2}\right\}$.

- b. On this equation we need to again start by getting rid of the parenthesis. Once we do that we can continue as normal

$$8u - 5 = (u - 4)(u - 2)$$

$$8u - 5 = u^2 - 6u + 8$$

$$-8u - 8 \quad -8u - 8$$

$$u^2 - 14u = -13$$

$$u^2 - 14u + \left(\frac{1}{2} \cdot 14\right)^2 = -13 + \left(\frac{1}{2} \cdot 14\right)^2$$

$$(u - 7)^2 = -13 + 49$$

$$(u - 7)^2 = 36$$

$$u - 7 = \pm\sqrt{36}$$

$$u = 7 \pm 6$$

Notice that we can actually perform the operations of + and - in the last step. Therefore, we must. So we have

$$u = 7 + 6$$

$$u = 7 - 6$$

$$= 13$$

$$= 1$$

So our solution set is $\{1, 13\}$.

The fact that we got rational numbers in the answer of the last example tells us that factoring could have been used to solve the quadratic in that problem. However, since the instructions told us to solve by completing the square, we had to solve using completing the square anyway.

10.2 Exercises

Complete the square on the following. Then factor accordingly.

1. $x^2 - 4x$

2. $x^2 + 6x$

3. $y^2 + 8y$

4. $x^2 - 10x$

5. $x^2 + 3x$

6. $y^2 - 5y$

7. $t^2 - \frac{1}{2}t$

8. $x^2 + \frac{1}{3}x$

9. $x^2 + \frac{2}{5}x$

10. $t^2 + \frac{4}{9}t$

Solve the equations by completing the square.

11. $x^2 - 4x - 5 = 0$

12. $x^2 + 6x + 8 = 0$

13. $y^2 + 8y - 1 = 0$

14. $x^2 - 10x - 3 = 0$

15. $x^2 + 3x + 1 = 0$

16. $y^2 - 5y - 4 = 0$

17. $t^2 - \frac{1}{2}t + 2 = 0$

18. $x^2 + \frac{1}{3}x - \frac{1}{9} = 0$

19. $x^2 + \frac{2}{5}x + \frac{1}{25} = 0$

20. $t^2 + \frac{4}{9}t + \frac{4}{81} = 0$

21. $2x^2 - 4x + 1 = 0$

22. $3x^2 - 6x + 2 = 0$

23. $3x^2 + 18x - 2 = 0$

26. $-3x^2 - 9x + 1 = 0$

29. $x^2 + x + 2 = 0$

32. $15t^2 + 5t = 2$

35. $\frac{1}{2}x^2 + 2x = 4$

38. $\frac{3}{4}y^2 + y = -1$

41. $28 - x - 2x^2 = 0$

44. $9x^2 = 1 + 9x$

47. $\frac{2x^2}{5} - \frac{x}{2} = 1$

49. $25(x-3)^2 - 36 = 0$

51. $2x(x-4) = 1$

53. $x(x+5) - 10(x+5) = 0$

55. $(3x+1)(2x-4) = 7$

57. $y(y-8) + 3y(y-1) = y$

24. $2x^2 + 6x - 9 = 0$

27. $-x^2 + 4x - 5 = 0$

30. $x^2 + 13 = 4x$

33. $r^2 + 3r = 8$

36. $\frac{1}{3}x^2 + 7 = x$

39. $2x^2 - 3x = -4$

42. $x^2 - 36x = 0$

45. $2y^2 + 3y + 1 = 0$

48. $(x-2)(x+9) = 10$

50. $(v-3)^2 = 250$

52. $x(3x-5) = x^2$

54. $(x+3)(2x+1) = -3$

56. $2y(y-18) + 3(y-18) = 0$

58. $x(2x-6) + x(x+5) = 0$

25. $-4x^2 + 8x = 0$

28. $-2x^2 - 2x = 0$

31. $12t^2 + 9t = 1$

34. $2x^2 - x + 5 = 0$

37. $\frac{2}{5}x^2 = 3x + 7$

40. $135 + 24u + u^2 = 0$

43. $2z^2 - 4z + 7 = 9$

46. $3t^2 = 2t - 5$