10.1 Solving Quadratic Equations by Factoring and Extracting Roots

Recall that at the end of chapter 4 we discussed solving polynomial equations. We want to expand upon one particular type of polynomial equation called the quadratic equation.

Definition: Quadratic Equation
A quadratic equation is an equation of the form $ax^2 + bx + c = 0$. When the equation is written with all terms on one side and zero on the other, the equation is in standard form.

We want to be able to solve all possible quadratic equations. The first method we can use is the method that we learned in chapter 4. That is, by factoring.

In order to solve by factoring we need the following property.

The Zero Product Property
If $a \cdot b = 0$ then $a = 0$ or $b = 0$.

Example 1:

Solve the following equations by factoring.

a. $y^2 + 6y = 0$ b. $2u^2 - 9u = -9$ c. $t + 24 = t(t + 6)$

Solution:

a. We first begin by factoring completely. Recall the first step in factoring is factor out the GCF. Then by the zero product property we set each factor to zero and solve. So we get

$$y^2 + 6y = 0$$
$$y(y + 6) = 0$$
$$y = 0 \quad y + 6 = 0$$
$$y = -6$$

So the solution set is $\{0, -6\}$.

b. In order to solve a quadratic by factoring we first need it in standard form. So we will begin by moving the 9 to get a zero on the one side. Then factor and set each factor to zero. We get

$$2u^2 - 9u = -9$$
$$2u^2 - 9u + 9 = 0$$
$$(2u - 3)(u - 3) = 0$$
$$2u - 3 = 0 \qquad u - 3 = 0$$
$$u = \tfrac{3}{2} \qquad u = 3$$

So the solution set is $\{\tfrac{3}{2}, 3\}$

c. Again we need to start by getting the equation into standard form. That means we need to multiply out the parenthesis and move everything to one side. Then we solve by factoring as before.

$$t + 24 = t(t + 6)$$
$$t + 24 = t^2 + 6t$$
$$-t - 24 \qquad -t - 24$$
$$0 = t^2 + 5t - 24$$
$$(t + 8)(t - 3) = 0$$
$$t = -8 \qquad t = 3$$

So the solution set is $\{-8, 3\}$.

We don't only want to solve quadratic equations but we also want to be able to get the equation based upon its solutions.

To do so we notice the following: If we solve the equation $(x - r_1)(x - r_2) = 0$ we get the solutions of $x = r_1, r_2$.

So this tells us that if we start with two solutions, we can simply plug them back into the formula $(x - r_1)(x - r_2) = 0$ to generate the equation.

<u>Example 2</u>:

Write the equation in standard form with integer coefficients that has solutions of the following

a. 3 and –1 b. $3\frac{1}{2}$ and $-\frac{1}{4}$

Solution:

a. We start by labeling $r_1 = 3$ and $r_2 = -1$. Now we insert them into the formula and multiply out to get

$$(x - r_1)(x - r_2) = 0$$
$$(x - 3)(x - (-1)) = 0$$
$$(x - 3)(x + 1) = 0$$
$$x^2 - 2x - 3 = 0$$

So the equation is $x^2 - 2x - 3 = 0$.

b. Again we label $r_1 = 3\frac{1}{2}$ and $r_2 = -\frac{1}{4}$. We insert them into the formula and multiply as before to get

$$(x - r_1)(x - r_2) = 0$$
$$\left(x - 3\tfrac{1}{2}\right)\left(x - \left(-\tfrac{1}{4}\right)\right) = 0$$
$$\left(x - \tfrac{7}{2}\right)\left(x + \tfrac{1}{4}\right) = 0$$
$$x^2 + \tfrac{1}{4}x - \tfrac{7}{2}x - \tfrac{7}{8} = 0$$

At this point we recall we wanted integer coefficients. Therefore we must clear the fractions. We choose to do this now since it is easier to combine like terms after we clear the fractions. So we multiply both sides by the LCD of 8 and continue

$$8\cdot\left(x^2+\tfrac{1}{4}x-\tfrac{7}{2}x-\tfrac{7}{8}\right)=(0)\cdot 8$$
$$8x^2+2x-28x-7=0$$
$$8x^2-26x-7=0$$

So the equation is $8x^2-26x-7=0$.

Everything we have done in this section is great, however, we know that not everything factors. So we want to start to build the tools for solving all quadratic equations. The first of these tools is called extracting roots and only works when a quadratic equation is in a very particular form.

First consider the equation $x^2=16$. Solving by factoring we get

$$x^2=16$$
$$x^2-16=0$$
$$(x-4)(x+4)=0$$
$$x=-4,\ 4$$

In order to simplify the two answers into one single expression we simply write

$$x=\pm 4$$

So from this we can see the following

Principle of Extracting Roots
If $u^2=a$ then $u=\pm\sqrt{a}$

So what this property states is, when we have a squared expression on one side of the equation, we can "square root" both sides to simplify. We simply need to remember to put the $\pm$ in front of the numerical side to show that we have two possible answers.

The real reason for the $\pm$ symbol goes back to the property that states $\sqrt{a^2}=|a|$. For simplicity we omitted the absolute value however, knowing how to solve absolute value equations we would really get

$$x^2=a^2$$
$$x=\sqrt{a^2}$$
$$x=|a|$$
$$x=-a,\ a$$
$$x=\pm a$$

Whichever way you choose to remember it, (either as just a rule or as the absolute value) just make sure that the $\pm$ is used.

<u>Example 3</u>:

Solve the following.

a. $x^2-32=0$ b. $(x+2)^2-25=0$ c. $4(y-2)^2+36=0$

Solution:

a. Since factoring can clearly not solve this equation we will apply the principle of extracting roots. To do this we must first get the x^2 isolated on one side. We then "take the square root of both sides" and attach the $\pm$ symbol.

$$x^2 - 32 = 0$$
$$x^2 = 32$$
$$x = \pm\sqrt{32}$$
$$x = \pm 4\sqrt{2}$$

So the solution set is $\{-4\sqrt{2},\ 4\sqrt{2}\}$.

b. Again we need to extract the roots to solve. First we need to isolate the portion of the equation which is squared. Then we extract the roots as follows

$$(x+2)^2 - 25 = 0$$
$$(x+2)^2 = 25$$
$$x + 2 = \pm\sqrt{25}$$
$$x + 2 = \pm 5$$

We still need to completely isolate the variable x so we have

$$x = -2 \pm 5$$

Now, recall the $\pm$ represents two possible cases, + and -. So since we can actually calculate –2 + 5 and –2 - 5 we need to split the $\pm$ symbol up and finish calculating the solutions. We have

$$x = -2 + 5 \qquad x = -2 - 5$$
$$x = 3 \qquad x = -7$$

So the solution set is $\{-7,\ 3\}$.

c. Finally, we again use extracting roots to solve.

$$4(y-2)^2 + 36 = 0$$
$$4(y-2)^2 = -36$$
$$(y-2)^2 = -9$$
$$y - 2 = \pm\sqrt{-9}$$
$$y = 2 \pm 3i$$

Recall, $\sqrt{-1} = i$. So, $\sqrt{-9} = \sqrt{-1} \cdot \sqrt{9} = 3i$

So we have two complex number solutions. The solution set is $\{2+3i,\ 2-3i\}$.

10.1 Exercises

Solve the following equations by factoring.

1. $(x+3)(x-4) = 0$
2. $(x+9)(x-8) = 0$
3. $x^2 - 17x + 42 = 0$
4. $x^2 - 7x - 18 = 0$
5. $6x^2 + 19x - 7 = 0$
6. $6x^2 - 5x + 1 = 0$
7. $x^2 - 7x = 0$
8. $2x^2 - 50 = 0$
9. $121 - 9x^2 = 0$

10. $x^2 + 14x = -49$
11. $7x^2 = 25x + 12$
12. $48 = 2x^2 + 4x$
13. $2x^2 = 128$
14. $2x^3 = x^2 + 21x$
15. $15x^3 = 11x^2 - 2x$
16. $6x^2 = 8x$
17. $x(x-5) = 6$
18. $x(6x+13) = 5$
19. $x(x-14) = 15$
20. $(x-6)(x+1) = 8$
21. $x + 3 = x(x+3)$
22. $t + 18 = t(t+8)$
23. $x^2 + (x+1)^2 = 313$
24. $x^2 + (x+1)^2 = 421$
25. $(x-4)(x-3) = 6$
26. $(x+6)(1-x) = 10$
27. $(x+4)^2 - 169 = 0$
28. $(x-10)^2 - 49 = 0$
29. $2x(3x+2) = 5 - 6x^2$
30. $(2x+1)(2x-1) + 4x^2 + 5x = 2$

Write the equation in standard form with integer coefficients that has solutions of the following.

31. 1 and 2
32. 2 and 3
33. 2 and –1
34. –3 and 2
35. –4 and 4
36. –6 and –7
37. –3 and –4
38. $\frac{1}{2}$ and $\frac{1}{3}$
39. $-\frac{1}{4}$ and $-\frac{1}{2}$
40. $\frac{3}{5}$ and $-\frac{2}{3}$
41. $-\frac{4}{5}$ and $\frac{1}{4}$
42. $\frac{1}{2}$ and 3
43. $-\frac{1}{8}$ and –1
44. $-\frac{2}{3}$ and $-\frac{1}{4}$
45. $-\frac{1}{3}$ and $-\frac{2}{5}$
46. $-\frac{3}{2}$ and $\frac{2}{3}$
47. $\sqrt{2}$ and $-\sqrt{2}$
48. $-3+\sqrt{2}$ and $-3-\sqrt{2}$
49. $2+\sqrt{3}$ and $2-\sqrt{3}$
50. i and $-i$

Solve the following.

51. $x^2 = 4$
52. $y^2 = 36$
53. $(x-1)^2 = 9$
54. $(x+2)^2 = 4$
55. $(x+3)^2 = 7$
56. $(x-4)^2 = 5$
57. $(x-3)^2 - 12 = 0$
58. $(x+1)^2 - 8 = 0$
59. $(x-2)^2 + 9 = 0$
60. $9(x-3)^2 + 36 = 0$
61. $3(x+7)^2 - 12 = 0$
62. $2(x+2)^2 - 16 = 0$
63. $1 - (x+1)^2 = 0$
64. $4 - (z-4)^2 = 0$
65. $3(x-2)^2 - 3 = 0$
66. $12 - (x+6)^2 = 12$
67. $25(x-3)^2 = 0$
68. $(x+4)^2 - 1 = 0$
69. $4(x+6)^2 + 16 = 0$
70. $4(y-2)^2 - 1 = 0$
71. $(z - \frac{1}{2})^2 - \frac{1}{4} = 0$
72. $(x+2)^2 + \frac{1}{4} = 0$
73. $9(x - \frac{1}{3})^2 - 1 = 0$
74. $(x - \frac{1}{5})^2 + 1 = 0$
75. $\frac{1}{2}(x-3)^2 = 6$
76. $\frac{1}{3}(x-3)^2 - 6 = 0$
77. $-\frac{2}{3}(x-4)^2 + 6 = 0$
78. $-\frac{2}{5}(y+2)^2 + 10 = 0$
79. $-\frac{1}{2}(x - \frac{1}{3})^2 - \frac{1}{9} = 0$
80. $-\frac{1}{3}(x - \frac{2}{3})^2 - \frac{1}{12} = 0$